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SECOND ORDER DYNAMICAL SYSTEMS TECHNIQUE FOR GENERAL VARIATIONAL INEQUALITIES

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Abstract. In this paper, we introduce and consider a new second order dynamical system for solving variational inequalities, which is second order boundary value problem. Using the forward finite difference schemes, we suggest some multi-step iterative methods for solving the variational inequalities. Convergence analysis is investigated under certain mild conditions. Since the variational inequalities are equivalent to the complementarity problems, our results can be used to develop new techniques for them. It is an interesting problem to compare these methods with other technique for solving variational inequalities and related optimizations for further research activities.

REFERENCES

[1] W.F. Ames: *Numerical Methods for Partial Differential Equations*, Academic Press, New York, 1992.
[2] Ashish, M. Rani and R. Chugh: *Julia sets and Mandelbrot sets in Noor orbit*, Appl. Math. Comput., **228**(2014), No. 1, 615-631.
[3] Ashish, R. Chugh and M. Rani: *Fractals and Chaos in Noor Orbit: a Four-Step Feedback Approach*, Lap Lambert Academic Publishing, Saarbrucken, Germany, 2021.
[4] Ashish, J. Cao and M.A. Noor: *Stabilization of fixed points in chaotic maps using Noor orbit with applications in cardiac arrhythmia*, J. Appl. Anal. Comput., **13**(2023), No. 5, 2452-2470.
[5] A. Barbagallo and S.G. Lo Bainco: *A random elastic traffic equilibrium problem via stochastic quasi-variational inequalities*, Commun. Nonlinear Sci. Numer. Simul., **131**(2024), 107798.
[6] A. Bnouhachem, M.A. Noor and T.M. Rassias: *Three-step iterative algorithms for mixed variational inequalities*, Appl. Math. Comput., **183**(2006), 436-446.
[7] C. Chairatsiripong and T. Thianwan: *Novel Noor iterations technique for solving nonlinear equations*, AIMS Math., **7**(2022), No. 6, 10958-10976.
[8] S.Y. Cho, A.A. Shahid, W. Nazeer and S.M. Kang: *Fixed point results for fractal generation in Noor orbit and s-convexity*, SpringerPlus, **5**(2006), Art. No. 1843, 16 pages.
[9] R.W. Cottle , J.-S. Pang and R. E. Stone: *The Linear Complementarity Problem*, SIAM, 2009 (CAM 60).
[10] G. Cristescu and L. Lupsa: *Non-Connected Convexities and Applications*, Kluwer Academic Publishers, Dordrecht, Holland, 2002.
[11] P. Dupuis and A. Nagurney: *Dynamical systems and variational inequalities*, Ann. Oper. Res., **44**(1993), 7-42.
[12] R. Glowinski, J.L. Lions and R. Tremolieres: *Numerical Analysis of Variational Inequalities*, North-Holland, Amsterdam, 1981.
[13] R. Glowinski and P. Le Tallec: *Augmented Lagrangian and Operator Splitting Methods in Nonlinear Mechanics*, SIAM, Philadelphia, Pennsylvania, USA, 1989.

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[14] A.G. Khan, M.A. Noor and K.I. Noor: *Dynamical systems for generla quasi variational inequalities*, Ann. Funct. Anal., **6**(2015), No. 1, 193-209.
[15] D. Kinderlehrer and G. Stampacchia: *An Introduction to Variational Inequalities and Their Applications*, SIAM, Philadelphia, PA, USA., 2000.
[16] G.M. Korpelevich: *The extragradient method for finding saddle points and other problems*, Ekonom. Mat. Met., **12**(1976), 747-756.
[17] Y.C. Kwuni, A.A. Shahid, W. Nazeer, S.I. Butt, M. Abbas and S.M. Kang: *Tricorns and multicorns in Noor orbit with s-convexity*, IEEE Access, **7**(2019), 25297-25304.
[18] J. Lions and G. Stampacchia: *Variational inequalities*, Comm. Pure Appl. Math., **20**(1967), 493-519.
[19] A. Nagurney and D. Zhang: *Projected Dynamical Systems and Variational Inequalities with Applications*, Kluwer Academic Publishers, Boston, Dordrecht, London 1996.
[20] K. Nammanee, M.A. Noor and S. Suantai: *Convergence criteria of modified Noor iterations with errors for asymptotically nonexpansive mappings*, J. Math. Anal. Appl., **314**(2006), 320-334.
[21] C.P. Niculescu and L.E. Persson: *Convex Functions and Their Applications*, Springer-Verlag, New York, 2018.
[22] M.A. Noor: On variational inequalities, Ph. D. Thesis, Brunel University, London, U. K. 1975.
[23] M.A. Noor: *The quasi-complementarity problem*, J. Math. Anal. Appl., **130**(1998), 344-353.
[24] M.A. Noor: *General variational inequalities*, Appl. Math. Lett., **1**(1988), 119-121.
[25] M.A. Noor: *Fixed point approach for complementaritty problems*, J. Math. Anal. Appl., **33**(1988), 437-448.
[26] M.A. Noor: *Variational inequalities for fuzzy mappings* (III), Fuzzy Sets and Systems, **110**(2000), No. 1, 101-108.
[27] M.A. Noor: *New approximation schemes for general variational inequalities*, J. Math. Anal. Appl., **251**(2000), 217-230.
[28] M.A. Noor: *A Wiener-Hopf dynamical system and variational inequalities*, New Zealand J. Math., **31**(2002), 173-182.
[29] M.A. Noor: *Some dvelopments in general variational inequalites*, Appl. Math. Comput., **152**(2004), 199-277.
[30] M.A. Noor: *Three-step iterative algorithms for multivalued quasi variational inclusions*, J. Math. Anal. Appl., **255**(2001), No. 2, 589-604.
[31] M.A. Noor: *Extended general variational inequalities*, Appl. Math. Lett., **22**(2009), 182-185.
[32] M.A. Noor: *Differentiable non-convex functions and general variational inequalities*, Appl. Math. Comput. **199**(2008), No. 2, 623-630.
[33] M.A. Noor and E.A. Al-Said: *Change of variable method for generalized complementarity problems*, J. Optim. Theory Appl., **100**(1999), No. 2, 389-395.
[34] M.A. Noor and K.I. Noor: *Dynamical system technique for solving quasi variational inequalities*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., **84**(2022), No. 4, 55-66.
[35] M.A. Noor and K.I. Noot: *Iterative schemes for solving general variational inequalities*, Differ. Equ. Appl., **15**(2023), No. 2, 113-134.
[36] M.A. Noor and K.I. Noor: *Some novel aspects and applications of Noor iterations and Noor orbits*, J. Adv. Math. Stud., **17**(2024), No. 3, 276-284.
[37] M.A. Noor, K.I. Noor and A.G. Khan: *Dynamical systems for quasi variational inequalities*, Ann. Funct. Anal., **6**(2015), No. 1, 193-209.
[38] M.A. Noor, K.I. Noor and A. Bnouhachem: *Some new iterative methods for variational inequalities*, Canad. J. Appl. Math., **3**(2021), No. 1, 1-17.
[39] M.A. Noor, K.I. Noor and M.T. Rassias: *New trends in general variational inequalities*, Acta Appl. Math., **170**(2020), No. 1, 981-1046.
[40] M.A. Noor, K.I. Noor and Th.M. Rassias: *Some aspects of variational inequalities*, J. Comput. Appl. Math., **47**(1993), 285-312.
[41] M.A. Noor and W. Oettli: *On general nonlinear complementarity problems and quasi equilibria*, Le Mathematiche, **49**(1994), 3131.
[42] S. Paimsang, D. Yambangwai and T. Thainwan: *A novel Noor iterative method of operators with property* (E) *as concerns convex programming applicable in signal recovery and polynomiography*, Math. Methods Appl. Sci., (2024), 1-18.
[43] M. Patriksson: *Nonlinear Programming and Variational Inequalities: A Unified Approach*, Kluwer Acadamic Publishers, Drodrecht, 1998.

[44] W. Phuengrattana and S. Suantai: *On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval*, J. Comput. Appl. Math., **235**(2011), No. 9, 3006-3014.
[45] K. Rattanaseeha, S. Imnang, P. Inkrong and T. Thianwan: *Novel Noor iterative methods for mixed type asymptotically nonexpansive mappings from the perspective of convex programming in hyperbolic spaces*, Int. J. Innov. Comput. Inform. Cont., **19**(2023), No. 6, 1717-1734.
[46] K. Sanaullah, S. Ullah and N.M. Aloraini: *A self adaptive three-step numerical scheme for variational inequalities*, Axioms, **13**(2024), Art. No. 57, 13 pages.
[47] G. Stampacchia: *Formes bilineaires coercitives sur les ensembles convexes*, C. R. Math. Acad. Sci. Paris, **258**(1964), 4413-4416.
[48] S. Suantai, M.A. Noor, K. Kankam and P. Cholamjiak: *Novel forward-backward algorithms for optimization and applications to compressive sensing and image inpainting*, Adv. Difference Equ., **2021**(2021), Art. No. 265, 22 pages.
[49] P. Tseng: *A modified forward-backward splitting method for maximal monotone mappings*, SIAM J. Control Optim., **38**(2000), 431-446.
[50] Y.S. Xia and J. Wang: *A recurrent neural network for solving linear projection equations*, Neural Networks, **13**(2000), 337-350.
[51] Y.S. Xia and J. Wang: *On the stability of globally projected dynamical systems*, J. Optim. Theory Appl., **106**(2000), 129-150.

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