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SOME NEW RESULTS FOR SET VALUED MAPPINGS DEFINED ON SETS ON METRIC SPACE WITH GRAPH

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Abstract. In the present paper, we define the notion of (A_φ, G) contraction mapping on subsets of a metric space involving a graph. The existence as well as uniqueness of fixed point of such contractions have been examined. Example in support of the results is also provided.

1. INTRODUCTION

In nonlinear analysis, the study of fixed points for set valued maps is very important. This theory has been applied in different fields such as control theory, convex optimization, differential inclusions and economics. Using the same line as the Banach contraction principle, Nadler [14], defining the notion of set valued contractions, proved that a set valued contraction mapping admits a fixed point on a complete metric space. Subsequently, Nadler's fixed point theorem has been extended by many researchers in different point of view.

In order to study the fixed point results in partially ordered sets Ran and Reurings [19] combined the Banach contraction principle and the Knaster-Tarski fixed point theorem for continuous functions. After that, Neito & Rodríguez-López ([16, 17]) extended the results of Ran and Reurings to the functions which were not necessarily continuous.

The two distinct branches of mathematics, i.e., fixed point theory and graph theory, were combined by Espinola and Kirk [11] to establish some fixed point results. In 2008, Jachymski [12], replaced the concept of partial ordering with that of graph theory. He restricted the contractive condition to the edges of the graph and hence gave fixed point theorems for metric space involving with a directed graph. Following Jachymski, Bojor [7] established some fixed point theorems for Reich type contractions on metric space with a graph. Furthermore, he obtained a fixed point theorem for Kannan mappings.

Some fixed point results on subgraphs of directed graphs were established by Aleomraninejad *et al.* [5]. Recently, there has been a significant development in this direction [3, 6, 9, 10, 15, 21].

Akram *et al.* [4] gave a characterization for a complete metric space in terms of fixed point property for A-contractions. In the current paper, we define the notion of (A_φ, G)

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contraction on closed and bounded subsets of a metric space involving the graph and obtain some new results for such contractions.

Suppose (X, d) be a metric space and $CB(X)$ be the collection of all nonempty closed and bounded subsets of X. If

$$H(A, B) = \max\Big\{\sup_{b\in B} d(b, A), \sup_{a\in A} d(a, B)\Big\}, \quad A, B \in CB(X),$$

where $d(a, B) = \inf_{b\in B} d(a, b)$. Then H is said to be a Hausdorff-Pompeiu metric induced by d and $(CB(X), H)$ forms a metric space.

A directed graph G is an ordered pair $(V(G), E(G))$, where $V(G)$ is the set of vertices and $E(G)$ is the set of edges. We consider G as $V(G) = X$ and the set $E(G)$ of its edges also contains all its loops. Also, we assume that G does not contain parallel edges.

The conversion of a graph G is denoted by G^{-1} and it is obtained from G by reversing the direction of edges of G. Also, the undirected graph is denoted by $\widetilde{G}$ and it is obtained from G by ignoring the direction of edges. We consider a directed graph G such that $E(G)$ is symmetric, so we get

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}).$$

If x, y are two vertices of G, then a path in G from x to y is a finite sequence $\{x_i\}_{i=0}^{n}$ of $n+1$ vertices such that $x_0 = x$, $x_n = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, \ldots, n$.

The graph G is said to be connected if there is a path between every pair of vertices in G. Further, G is weakly connected if $\widetilde{G}$ is connected.

In $V(G)$, the relation R is defined by the following rule:

For $a, b \in V(G)$, aRb if and only if there is a path from a to b in G.

If the set of edges of the graph G is symmetric and $x \in V(G)$, then the subgraph G_x that contains all edges and vertices contained in some path which begins at x is called the component of G containing x. We denote this by $V(G_x) = [x]_G$, where $[x]_G$ is the equivalence class of R.

In this paper, our main goal is to consider the mapping $T\colon CB(X) \to CB(X)$ instead of $T\colon X \to X$ or $T\colon X \to CB(X)$ to study fixed point results of set valued mappings with set valued domain. The investigation of set valued mappings with set valued domains were initiated by [13]. Further, to produce more general results in nonlinear analysis, consideration of mappings of the form $T\colon CB(X) \to CB(X)$ was carried out by Debnath [8]. In the similar setting, recently, more fixed point results have been investigated [1, 2, 9, 15].

2. PRELIMINARIES

In $CB(X)$ we define the relation R as:

$$\{A, B \in CB(X)\colon\ ARB \text{ if and only if, there is a path between } A \text{ and } B\}.$$

For $A \in CB(X)$, the equivalence class $[A]_G$ of R is defined as:

$$[A]_G = \{B \subseteq X : ARB\}.$$

Suppose $A, B \subset X$ $(A, B \neq \phi)$. Then, by $(A, B) \subset E(G)$, we mean that 'there is an edge between A and B' , i.e., there is an edge between some $a \in A$ and $b \in B$. Also, by ARB, where R is a relation on $CB(X)$, we mean that 'there is a path between A and B' , i.e., there is a path between some $a \in A$ and $b \in B$.

For $T\colon CB(X) \to CB(X)$, we define the set X_T as:

$$X_T = \{A \in CB(X) : (A, T(A)) \subset E(G)\}.$$

Now, we list some important definitions which are useful in our main results. In the following $\mathbb{R}_+$ and $\mathbb{N}$ denote the set of all non negative real numbers and the set of all natural numbers respectively.

Definition 2.1 ([4]). Suppose $\mathbb{R}_+$ is the set of all non negative real numbers and A be the collection of all functions $\alpha\colon \mathbb{R}_+^3 \to \mathbb{R}_+$ which satisfy the conditions given below:

(i) α is continuous on $\mathbb{R}_+^3$ (with respect to the Euclidean metric on $\mathbb{R}_+^3$).

(ii) $a \le kb$ for some $k \in [0, 1)$ whenever $a \le \alpha(a, b, b)$ or $a \le \alpha(b, a, b)$ or $a \le \alpha(b, b, a)$, for all a, b.

Definition 2.2 ([4]). Suppose (X, d) is a metric space and T is a self map on X. T is called a A-contraction if

$$d(Tx, Ty) \le \alpha(d(x, y), d(x, Tx), d(y, Ty)), \quad \forall x, y \in X$$

and some $\alpha \in A$.

Definition 2.3 ([12]). A self mapping T on a metric space (X, d) is called a Banach G-contraction or G contraction if it satisfies the conditions given below:

(i) There is an edge between x and y implies there is an edge between $T(x)$ and $T(y)$, for all $x, y \in X$ i.e., $(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$.

(ii) T decreases weights of edges of G it means there exists an $\eta \in (0, 1)$ such that

$$(x, y) \in E(G) \Rightarrow d(Tx, Ty) \le \eta d(x, y), \quad \forall x, y \in X.$$

Definition 2.4 ([20]). Consider the class of functions $\Phi = \{\varphi | \varphi : \mathbb{R}_+ \to \mathbb{R}_+\}$, which satisfies the following assertions:

(i) $t_1 \le t_2$ implies $\varphi(t_1) \le \varphi(t_2)$;

(ii) $(\varphi^n(t))_{n\in\mathbb{N}}$ converges to 0, for all $t > 0$;

(iii) $\sum \varphi^n(t)$ converges, for all $t > 0$.

If conditions (i)-(ii) hold then φ is called a comparison function, and, if the comparison function satisfies the condition (iii) also, then φ is called a strong comparison function.

Remark 2.1 ([20]). Any strong comparison function is a comparison function.

Remark 2.2 ([20]). If $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function, then $\varphi(t) < t$, for all $t > 0$, $\varphi(0) = 0$ and φ is right continuous at 0.

Following Petrusel and Rus [18], Jachymski [12] gave the definition of Picard operator on metric space.

Definition 2.5 ([12]). Let (X, d) be a metric space and f is a self map on X. f is called a Picard operator if f has a unique fixed point x_0 and $\lim\limits_{n\to\infty} f^n x = x_0$, for all $x \in X$.

3. MAIN RESULTS

Now we discuss our main results, by defining (A_φ, G) contraction for set valued mapping with set valued domain. Also, we give some definitions which are used in our main results.

Definition 3.1. Suppose $\mathbb{R}_+$ is the set of all non-negative real numbers and A_φ is the collection of all functions $\alpha\colon \mathbb{R}^3_+ \to \mathbb{R}_+$ which satisfy the following conditions:

(i) α is continuous on $\mathbb{R}^3_+$ (with respect to the Euclidean metric on $\mathbb{R}^3_+$),

(ii) for all $u, v \in \mathbb{R}_+$, $u \leq \alpha(u, v, v)$ or $u \leq \alpha(v, u, v)$ or $u \leq \alpha(v, v, u)$, then $u \leq \varphi(v)$, where φ is a strong comparison function.

In this definition, if we take $\varphi(t) = kt$ as $k \in (0, 1)$, for all $t > 0$, then we obtain $\alpha \in A$.

Definition 3.2. Suppose (X, d) is a metric space and T is a self map on X. T is said to be a (A_φ, G)-contraction, if it satisfies the conditions given below:

(i) There is an edge between x and y implies there is edge an between $T(x)$ and $T(y)$, for all $x, y \in X$ i.e., $(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$,

(ii) there exists some $\alpha \in A_\varphi$ such that

$$d(Tx, Ty) \leq \alpha(d(x, y), d(x, Tx), d(y, Ty)),$$

for each $(x, y) \in E(G)$.

Remark 3.1. Let (X, d) be a metric space involving a graph G and $T\colon X \to X$ be a (A_φ, G)-contraction. If there exists $x_0 \in X$ such that $Tx_0 \in [x_0]_{\widetilde{G}}$, then

(i) T is both a $(A_\varphi, \widetilde{G})$ contraction and a (A_φ, G^{-1}) contraction.

(ii) $[x_0]_{\widetilde{G}}$ is T-invariant and $T|_{[x_0]_{\widetilde{G}}}$ is a $(A_\varphi, \widetilde{G}_{x_0})$ contraction.

Definition 3.3. The set valued mapping $T\colon CB(X) \to CB(X)$ is said to be a (A_φ, G) contraction if for all $A, B \in CB(X)$, the following conditions hold:

(i) T preserves edges of G i.e. $(A, B) \subset E(G) \Rightarrow (T(A), T(B)) \subset E(G)$.

(ii) T preserves paths of G i.e. $ARB \Rightarrow T(A)RT(B)$.

(iii) There exists some $\alpha \in A_\varphi$ such that

$$H(T(A), T(B)) \leq \alpha(H(A, B), H(A, T(A)), H(B, T(B))),$$

for each $(A, B) \subset E(G)$.

Remark 3.2. If the mapping $T\colon CB(X) \to CB(X)$ is a (A_φ, G) contraction, then T is both $(A_\varphi, \widetilde{G})$ contraction and a (A_φ, G^{-1}) contraction.

Definition 3.4. We say that the set $A \in CB(X)$ is a fixed point of $T\colon CB(X) \to CB(X)$ if $T(A) = A$.

Definition 3.5 ([1])**.** A graph G is said to have property (P^*) if for any sequence $\{X_n\}_{n\geq 0}$ in $CB(X)$ with $X_n \to X$ as $n \to \infty$, existence of an edge between X_n and X_{n+1} for $n \in \mathbb{N}$, implies that there is a subsequence $\{X_{n_k}\}$ of $\{X_n\}$ with an edge between X_{n_k} and X for $k \in \mathbb{N}$.

Definition 3.6. Let $(CB(X), H)$ be a metric space and $T\colon CB(X) \to CB(X)$ is a set valued map on X. We say that T is a Picard operator if T has a unique fixed point A_0 and $\lim\limits_{n\to\infty} T^n(A) = A_0$, for all $A \in CB(X)$.

Definition 3.7. We say that a set valued mapping $T\colon CB(X) \to CB(X)$ is orbitally G-continuous if, for any sequence $(n_p)_{p\in\mathbb{N}}$ of positive integers, $T^{n_p}(U) \to V$, $(T^{n_p}(U), T^{n_{p+1}}(U)) \subset E(G)$ imply $\lim_{p\to\infty} T(T^{n_p}(U)) = T(V)$, for all $U, V \in CB(X)$.

Lemma 3.1. *Let (X,d) be a metric space involving a graph G and $T\colon CB(X) \to CB(X)$ be a (A_φ, G) contraction. Then given $A_0 \in X_T$, there exists $r(A, T(A_0)) \geq 0$ such that,*

$$H(T^n(A_0), T^{n+1}(A_0)) \leq \varphi^n(r(A_0, T(A_0))),$$

for all $n \in \mathbb{N}$, where $r(A_0, T(A_0)) = H(A_0, T(A_0))$.

Proof. Let $A_0 \in X_T$. By the definition of X_T we have, $(A_0, T(A_0)) \subset E(G)$. Since $T\colon CB(X) \to CB(X)$ is (A_φ, G) contraction, so

$$(A_0, T(A_0)) \subset E(G) \Rightarrow (T(A_0), T^2(A_0)) \subset E(G).$$

Continuing in this way we get $(T^n(A_0), T^{n+1}(A_0)) \subset E(G)$, for all $n \in \mathbb{N}$.

Thus from the given contraction condition and using the definition of α, we get

$$\begin{aligned} H(T^n(A_0), T^{n+1}(A_0)) &\leq \alpha(H(T^{n-1}(A_0), T^n(A_0)), H(T^{n-1}(A_0), T(T^{n-1}(A_0))), \\ &\quad H(T^n(A_0), T(T^n(A_0))) \\ &= \alpha(H(T^{n-1}(A_0), T^n(A_0)), H(T^{n-1}(A_0), T^n(A_0)), \\ &\quad H(T^n(A_0), T^{n+1}(A_0))) \\ \Rightarrow H(T^n(A_0), T^{n+1}(A_0)) &\leq \varphi(H(T^{n-1}(A_0), T^n(A_0))). \end{aligned} \tag{3.1}$$

In this way, for all $n \in \mathbb{N}$, we obtain

$$\begin{aligned} H(T^n(A_0), T^{n+1}(A_0)) &\leq \varphi(H(T^{n-1}(A_0), T^n(A_0))) \\ &\leq \varphi[\varphi(H(T^{n-2}(A_0), T^{n-1}(A_0)))] = \varphi^2(H(T^{n-2}(A_0), T^{n-1}(A_0))) \\ &\vdots \\ &\leq \varphi^n(H(A_0), T(A_0))) = \varphi^n(r(A_0, T(A_0))). \end{aligned}$$

Thus,

$$H(T^n(A_0), T^{n+1}(A_0)) \leq \varphi^n(r(A_0, T(A_0))), \quad \forall n \in \mathbb{N},$$

where $r(A_0, T(A_0)) = H(A_0, T(A_0))$. □

Theorem 3.1. *Let $(CB(X), H)$ be a complete metric space involving a graph G and $T\colon CB(X) \to CB(X)$ be a set valued mapping with set valued domain, which satisfies the conditions given below:*

(i) G is weakly connected and satisfies the property (P^*);
(ii) T is $(A_\varphi, \widetilde{G})$ contraction;
(iii) X_T is nonempty.

Then T is a Picard Operator (PO).

Proof. Let $A_0 \in X_T$, then $(T^n(A_0), T^{n+1}(A_0)) \subset E(G)$, for all $n \in \mathbb{N}$.

Thus, Lemma 3.1, gives

$$H(T^n(A_0), T^{n+1}(A_0)) \leq \varphi^n(r(A_0, T(A_0))).$$

Also $r(A_0, T(A_0)) = H(A_0, T(A_0)) \geq 0$. Therefore by using the definition 2.4(ii), we have $\lim_{n\to\infty} \varphi^n(H(A_0, T(A_0))) = 0$.

Now, for a given $\varepsilon > 0$, there is a positive integer n_0 such that for all $n \geq n_0$

$$\varphi^n(H(A_0, T(A_0))) < \varepsilon - \varphi(\varepsilon) \Rightarrow \varphi^n(r(A_0, T(A_0))) < \varepsilon - \varphi(\varepsilon).$$

Hence

$$H(T^n(A_0), T^{n+1}(A_0)) < \varepsilon - \varphi(\varepsilon), \quad \forall n \geq n_0. \tag{3.2}$$

Now for any $m, n \in \mathbb{N}$, with $m > n \geq n_0$, we claim that

$$H(T^n(A_0), T^m(A_0)) < \varepsilon. \tag{3.3}$$

We prove the inequality 3.3 by induction on m.

Using 3.2, we conclude that the inequality 3.3 holds for $m = n + 1$. Let us assume that 3.3 holds for $m = k$. That is, $H(T^n(A_0), T^k(A_0)) < \varepsilon$, so that for $m = k + 1$, we have

$$\begin{aligned} H(T^n(A_0), T^m(A_0)) &= H(T^n(A_0), T^{k+1}(A_0)) \\ &\leq H(T^n(A_0), T^{n+1}(A_0)) + H(T^{n+1}(A_0), T^{k+1}(A_0)) \\ &< \varepsilon - \varphi(\varepsilon) + \varphi(H(T^n(A_0), T^k(A_0))) \text{ [using 3.1 and 3.2]} \\ &< \varepsilon - \varphi(\varepsilon) + \varphi(\varepsilon) = \varepsilon. \end{aligned}$$

Hence by induction on m, we conclude that 3.3 holds for $m > n \geq n_0$. Thus $\{T^n(A_0)\}$ is a Cauchy sequence. Being $(CB(X), H)$ is complete, the Cauchy sequence $\{T^n(A_0)\}$ converges to A, for some $A \in CB(X)$.

Next, we show that A is a fixed point of T. Now, for all $n \in \mathbb{N}$, $(T^n(A_0), T^{n+1}(A_0)) \subset E(G)$ and $T^n(A_0) \to A$. By property (P^*), there is a subsequence $\{T^{n_p}(A_0)\}$ of $\{T^n(A_0)\}$ such that $(T^{n_p}(A_0), A) \subset E(G)$ for each $p \in \mathbb{N}$. Thus, $(T^{n_p}(A_0), A) \subset E(\widetilde{G})$ for each $p \in \mathbb{N}$. Since every subsequence of a convergent sequence is convergent and has the same limit. Therefore $\lim_{p\to\infty} T^{n_p}(A_0) = A$.

Hence,

$$H(T^{n_p+1}(A_0), T(A)) \leq \alpha(H(T^{n_p}(A_0), A), H(T^{n_p}(A_0), T^{n_p+1}(A_0)), H(A, T(A))).$$

Taking $p \to \infty$ in the above inequality, we get

$$H(A, T(A)) \leq \alpha(0, 0, H(A, T(A))).$$

From the definition of α, we get $H(A, T(A)) \leq \varphi(0) = 0$. Thus $A = T(A)$.

Finally, we show that A is a unique fixed point of T. Suppose $T^n(A_0) \to B$ and B is an another fixed point of T so that $A \neq B$. $T^n(A_0) \to A$, $T^n(A_0) \to B$ and $(T^n(A_0), T^{n+1}(A_0)) \subset E(G)$, for all $n \in \mathbb{N}$. Also G satisfies the property (P^*), therefore there is a subsequence $\{T^{n_p}(A_0)\}$ of $\{T^n(A_0)\}$ such that $(T^{n_p}(A_0), A) \subset E(G)$ and $(T^{n_p}(A_0), B) \subset E(G)$ for each $p \in \mathbb{N}$. Furthermore, G is weakly connected $(A, B) \subset E(\widetilde{G})$. So, $(T^{n_p}(A_0), A) \subset E(\widetilde{G})$ and $(T^{n_p}(A_0), B) \subset E(\widetilde{G})$, for each $p \in \mathbb{N}$.

Now,

$$\begin{aligned} H(A, B) = H(T(A), T(B)) &\leq \alpha(H(A, B), H(A, T(A)), H(B, T(B))) \\ &= \alpha(H(A, B), 0, 0). \end{aligned}$$

By the definition of α, we have $H(A, B) \leq \varphi(0) = 0$. Thus $A = B$ which proves the uniqueness of A. Hence T is a PO. □

Next, we prove the following theorem using the definition of orbitally G-continuous instead of the property (P^*).

Theorem 3.2. *Let $(CB(X), H)$ be a complete metric space involving a graph G and $T\colon CB(X) \to CB(X)$ be a set valued map with set valued domain. Assume that the following conditions hold:*

(i) G is weakly connected;
(ii) T is $(A_\varphi, \widetilde{G})$ contraction and orbitally G-continuous;
(iii) $X_T \neq \phi$.

Then T is a PO.

Proof. Let $A_0 \in X_T$, so $(T^n(A_0), T^{n+1}(A_0)) \subset E(G)$, for all $n \in \mathbb{N}$.

Now, using Theorem 3.1, we have $\{T^n(A_0)\}$ is a Cauchy sequence. Being $(CB(X), H)$ is complete, the Cauchy sequence $\{T^n(A_0)\}$ converges to A for some $A \in CB(X)$. Because $(T^n(A_0), T^{n+1}(A_0)) \subset E(G)$, for all $n \in \mathbb{N}$ and by assumption T is orbitally G-continuous, so $\lim_{n\to\infty} T(T^n(A_0)) = T(A)$. Now,

$$H(T^n(A_0), T^{n+1}(A_0)) \leq \alpha(H(T^{n-1}(A_0), T^n(A_0)), H(T^{n-1}(A_0), T^n(A_0)),$$
$$H(T^n A_0, T^{n+1}(A_0))).$$

Taking $n \to \infty$, we get

$$H(A, T(A)) \leq \alpha(0, 0, H(A, T(A)).$$

Therefore from the definition of α, we get $H(A, T(A)) \leq \varphi(O) = 0$ i.e. $(H(A, T(A)) = 0$. Thus $A = T(A)$. To show A is a unique fixed point of T, let $T(B) = B$ so that $A \neq B$. If we use the same procedure as in the Theorem 3.1, then we get $A = B$ □

Corollary 3.1. *Let $(CB(X), H)$ be a complete metric space involving a graph G. Also, $T\colon CB(X) \to CB(X)$ be an edge-preserving and X_T be a nonempty set, which satisfies the conditions given below:*

(i) G is weakly connected and satisfies the property (P^*);
(ii) there exists some $\alpha \in A$ such that

$$H(T(A), T(B)) \leq \alpha(H(A, B), H(A, T(A)), H(B, (T(B)),$$

for each $(A, B) \subset E(\widetilde{G})$.

Then T is a PO.

Corollary 3.2. *Let $(CB(X), H)$ be a complete metric space involving a graph G. Also, $T\colon CB(X) \to CB(X)$ be an edge-preserving and X_T be a nonempty set, which satisfies the conditions given below:*

(i) G is weakly connected ;
(ii) T is orbitally G-continuous;
(iii) there exists some $\alpha \in A$ such that

$$H(T(A), T(B)) \leq \alpha(H(A, B), H(A, T(A)), H(B, (T(B)),$$

for each $(A, B) \subset E(\widetilde{G})$.

Then T is a PO.

Our next example verifies all conditions of Theorem 3.1 and Theorem 3.2.

Example 3.1. Let $X = \{1,2,3,4\}$ and $\widetilde{G}$ be a graph with $V(\widetilde{G}) = X$ and

$$E(\widetilde{G}) = \{(1,1),(2,2),(3,3),(4,4),(1,2),(1,3),(1,4),(2,3),(2,4),(4,3)\}.$$

On $V(\widetilde{G})$, define the metric $d\colon X \times X \to \mathbb{R}^+$ as $d(x,x) = 0$, for all $x \in X$,

$$\begin{aligned} d(3,4) &= d(4,3) = \frac{1}{n+1}, \\ d(1,3) &= d(3,1) = d(1,4) = d(4,1) = d(2,3) = d(3,2) = d(1,2) = d(2,1) \\ &= d(2,4) = d(4,2) = \frac{n+1}{n+2}. \end{aligned}$$

Moreover, the Hausdorff-Pompeiu metric H is defined by

$$H(A,B) = \begin{cases} \dfrac{1}{n+1}, & \text{if } A,B \subseteq \{3,4\} \text{ with A} \neq B \\ \dfrac{n+1}{n+2}, & \text{if } A \text{ or } B \text{ (or both)} \nsubseteq \{3,4\} \text{ with } A \neq B \\ 0, & \text{if } A = B. \end{cases}$$

Let $T\colon CB(X) \to CB(X)$ as follows:

$$T(U) = \begin{cases} \{3\}, & \text{if } U \subseteq \{3,4\} \\ \{4\}, & \text{if } U \nsubseteq \{3,4\}. \end{cases}$$

It is clear that X_T is nonempty. And for all $A, B \in CB(X)$,

$$(A,B) \subset E(G) \Rightarrow (T(A),T(B)) \subset E(G).$$

Also, $ARB \Rightarrow T(A)RT(B)$.

Now, for all $A, B \in CB(X)$, we consider the following cases:

(a) If $A, B \subseteq \{3,4\}$, then we have $H(T(A),T(B) = H(\{3\},\{3\}) = 0$.

(b) If $A \nsubseteq \{3,4\}$, and $B \subseteq \{3,4\}$, we have $H(T(A),T(B) = H(\{4\},\{3\}) = \dfrac{1}{n+1}$. Since

$$\frac{1}{n+1} \leq \alpha\Big(\frac{n+1}{n+2}, \frac{n+1}{n+2}, \frac{1}{n+1}\Big) \Rightarrow H(T(A),T(B)) \leq \alpha(H(A,B),H(A,T(A)),H(B,T(B))).$$

(c) If $A, B \nsubseteq \{3,4\}$ we have $(H(T(A),T(B) = H(\{4\},\{4\}) = 0$.

Obviously, in the above three cases we see that Definition 3.3 is satisfied. Hence for all $A, B \in CB(X)$, Definition 3.3 is satisfied. So, T is a $(A_\varphi, \widetilde{G})$ contraction. Thus, all conditions of Theorem 3.1 and Theorem 3.2 are satisfied, where $\varphi(t) = \dfrac{3t}{4}$. Moreover, $\{3\}$ is a unique fixed point of T.

Conclusion. In this paper, we introduced the notion of (A_φ, G) contraction on closed and bounded subsets of a metric space and proved some fixed point results for set valued mappings with set valued domain. Our results, eventually, would extend and generalize the existing results in similar context. In a few of our results, we have used some conditions on the underlying graph G that it should be weakly connected. It would be an interesting topic for future study that if the condition of weak connectivity can be relaxed.

REFERENCES

[1] M. Abbas, M.R. Alfuraidan, A.R. Khan and T. Nazir: *Fixed point results for set-contractions on metric spaces with a directed graph*, Fixed Point Theory Appl., 2015, 2015:14, 9 pages.

[2] M. Abbas, T. Nazir, T.A. Lampert and S. Radenović: *Common fixed points of set-valued F-contraction mappings on domain of sets endowed with directed graph*, Comput. Appl. Math., **36**(2016), No. 4, 1607-1622.

[3] M. Abbas, T. Nazir, B.Z. Popović and S. Radenović: *On weakly commuting set valued mappings on a domain of sets endowed with directed graph*, Results Math., **71**(2017), No. 3-4, 1277-1295.

[4] M. Akram, A.A. Zafar and A.A. Siddiqui: *A general class of contractions: A-contractions*, Novi Sad J. Math., **38**(2008), 25-33.

[5] S.M.A. Aleomraninejad, S. Rezapour and N. Shahzad: *Some fixed point results on a metric space with a graph*, Topology Appl., **159**(2012), 659-663.

[6] M. Azam and M. Mehmood, T. Došenović and S. Radenović: *Coincidence point of L-fuzzy sets endowed with graph*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM, (2017), 1-17. doi: 10.1007/s13398-017-0398-3

[7] F. Bojor: *Fixed points of Kannan mappings in metric spaces endowed with a graph*, An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat., **20**(2012), No. 1, 31-40.

[8] P. Debnath: *Fixed points of contractive set valued mappings with set valued domains on a metric space with graph*, TWMS J. Appl. Eng. Math., **4**(2014), No. 2, 169-174.

[9] P. Debnath, B.S. Choudhury and M. Neog: *Fixed set of set valued mappings with set valued domain in terms of start set on a metric space with a graph*, Fixed Point Theory Appl., 2017, 2017:5, 8 pages.

[10] D. Eshi, P.K. Das, and P. Debnath: *Coupled coincidence and coupled common fixed point theorems on a metric space with a graph*, Fixed Point Theory Appl., 2016, 2016:37, 14 pages.

[11] R. Espinola and W.A. Kirk: *Fixed point theorems in R-trees with applications to graph theory*, Topology Appl., **153**(2006), 1046-1055.

[12] J. Jachymski: *The contraction principle for mappings on a metric space with a graph*, Proc. Amer. Math. Soc., **136**(2008), No. 4, 1359-1373.

[13] J. Jachymski, L. Gajek and P. Pokarowski: *The Tarski-Kantorovitch principle and the theory of iterated function systems*, Bull. Aust. Math. Soc., **61**(2000), 247-261.

[14] S.B. Nadler: *Multi-valued contraction mappings*, Pacific J. Math., **30**(1969), No. 2, 475-488.

[15] M. Neog and P. Debnath: *Fixed points of set valued mappings in terms of start point on a metric space endowed with a directed graph*, Mathematics, **5**(2017), No. 2, 7 pages. doi: 10.3390/math5020024

[16] J.J. Nieto and R. Rodríguez-López: *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order, **22**(2005), No. 3, 223-239.

[17] J.J. Nieto and R. Rodríguez-López: *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations*, Acta Math. Sin. (Engl. Ser.), **23**(2007), No. 12, 2205-2212.

[18] A. Petruşel and I.A. Rus: *Fixed point theorems in ordered L-spaces*, Proc. Amer. Math. Soc., **134**(2006), 411-418.

[19] A.C.M. Ran and M.C.B. Reurings: *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., **132**(2004), No. 5, 1435-1443.

[20] I.A. Rus, A. Petruşel G. and Petrusel: *Fixed Point Theory*, Cluj University Press, 2008.

[21] S. Shukla, S. Radenović and C. Vetro: *Graphical metric space: a generalized setting in fixed point theory*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM, **111**(2017), No. 3, 641-655.

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