GENERALIZED ZETA FUNCTIONS

TIAN-XIAO HE

Dedicated to Professor Leetsch C. Hsu on the Occasion of his 90th Birthday

Abstract. We present here a wide class of generalized zeta function in terms of the generalized Möbius functions and its properties.

1. INTRODUCTION

For any integer $z \in \mathbb{C}$, a Fleck-type generalized Möbius function (cf. [3] or [7]) of order $z$ can be defined by

$$
\mu_z(n) := \Pi_p(-1)^{e_p(n)} \left( \frac{z}{e_p(n)} \right)
$$

(1.1)

for any $n \in \mathbb{N}$, where $p$ runs through all the prime divisors of $n$, and $e_p(n) = \text{ord}_p(n)$ denotes the highest power $k$ of $p$ such that $p^k$ divides $n$. Obviously, $\mu_1(n) = \mu(n)$, $n \in \mathbb{N}$, is the classical Möbius function: $\mu(1) = 1$; if $n$ is not square free then $\mu(n) = 0$; if $n$ is square free and if $q$ is the number of distinct primes dividing $n$, then $\mu(n) = (-1)^q$. In addition, (1.1) implies

$$
\mu_0 = \Pi_{p|n}(-1)^{e_p(n)} \left( \frac{0}{e_p(n)} \right) = \begin{cases} 1, & n = 1, \\ 0, & n > 1, \end{cases}
$$

and

$$
\mu_{-1} = \Pi_p(-1)^{e_p(n)} \left( \frac{-1}{e_p(n)} \right) = \Pi_{p|n} \frac{(e_p(n))!}{(e_p(n))!} = 1.
$$

It is easy to verify that for each complex number $\alpha$, $\mu_\alpha$ is a multiplicative function, but is not complete multiplicative except $\mu_0$, which is complete multiplicative. The generalized zeta function, denoted by $\xi_z$, is defined accordingly by

$$
\xi_z(s) = \frac{1}{\sum_{n \geq 1} \frac{\mu_z(n)}{n^s}},
$$

(1.2)
where $z \in \mathbb{C}$. Hence, $\xi_1 = \xi$ the classical zeta function. And $\xi_0 = 1$.

**Remark 1.1.** $\xi_z(s)$ defined in (1.2) can be extended to $\mathbb{C}$. However, throughout this note, we do not consider the zero points of $\xi_1(s)$ in its domain, i.e., the points at which $\sum_{n \geq 1} \frac{\mu_1(n)}{n^s}$ diverges to infinite.

In this note, we will show that the set of functions $\xi_\alpha (\alpha \in \mathbb{C})$ forms an Abelian group with the Dirichlet series multiplication followed by a number of applications.

## 2. GENERALIZED ZETA FUNCTION GROUP

We now recall the definition of the Dirichlet product (or convolution) of two arithmetic functions $f$ and $g$ (cf. [1], [2]).

**Definition 2.1.** Given two arithmetic functions $f$ and $g$, the Dirichlet (convolution) product $f \ast g$ is again an arithmetic function which is defined by

$$ (f \ast g)(n) := \sum_{d \mid n} f(d)g \left( \frac{n}{d} \right) = \sum_{d \mid n} f \left( \frac{n}{d} \right) g(d), \quad (2.1) $$

where the summations are taken over all positive divisors $d$ of $n$.

**Definition 2.2.** Denote $M := \{ \mu_z : z \in \mathbb{C} \}$, where $\mathbb{C}$ denotes the set of complex numbers. We call $M$ the set of generalized Möbius functions of complex order. The set, denoted by $N$, of the corresponding nonzero generalized zeta functions of complex order is defined by $N := \{ \xi_z : z \in \mathbb{C} \}$, where $\xi_z$ are presented in (1.2).

From [2] and [5], $(M, \ast)$ forms an Abelian group with identity element $\mu_0$ under the operation $\ast : M \times M \mapsto M$, $\mu_\alpha \ast \mu_\beta = \mu_{\alpha+\beta}$, where $\alpha, \beta \in \mathbb{C}$.

**Lemma 2.1.** For any given $\alpha, \beta \in \mathbb{C}$, we define $\cdot : N \times N \mapsto N$ by

$$ \xi_\alpha \cdot \xi_\beta = \xi_\gamma $$

for some $\gamma \in \mathbb{C}$ if $\frac{1}{\xi_\alpha} \ast \frac{1}{\xi_\beta} = \frac{1}{\xi_\gamma}$, where $\ast$ is the regular Dirichlet product of Dirichlet series (cf. (2.1) and [8] for more details). Thus, we have

$$ \xi_\alpha \cdot \xi_\beta = \xi_{\alpha+\beta}. $$
Proof. By writing \(1/\xi_\alpha(s) = \sum_{n \geq 1} \frac{\mu_\alpha(n)}{n^s}\), and \(1/\xi_\beta(s) = \sum_{n \geq 1} \frac{\mu_\beta(n)}{n^s}\), we obtain

\[
1/(\xi_\alpha \xi_\beta)(s) = \sum_{n \geq 1} \left( \sum_{d|n} \mu_\alpha(d) \mu_\beta \left( \frac{n}{d} \right) \right) /n^s = \sum_{n \geq 1} (\mu_\alpha \ast \mu_\beta) /n^s
\]

\[
= \sum_{n \geq 1} \frac{\mu_{\alpha+\beta}}{n^s} = 1/\xi_{\alpha+\beta}(s),
\]

which completes the proof. \(\square\)

We now ready to show \((N, \cdot)\) is an Abelian group.

**Theorem 2.1.** Let \(\cdot\) be the operation define in Lemma 2.1. Then \((N, \cdot)\) is an Abelian group with identity element \(\xi_0 = 1\).

**Proof.** From Lemma 2.1, we see that \(N\) is closed respect to the operation \(\cdot\). Moreover, for any \(\alpha\) and \(\beta\) \(\in\mathbb{C}\), we have

\[\xi_\alpha \cdot \xi_\beta = \xi_{\alpha+\beta} = \xi_\beta \cdot \xi_\alpha.\]

And for any \(\alpha, \beta\) and \(\gamma\) \(\in\mathbb{C}\),

\[(\xi_\alpha \cdot \xi_\beta) \cdot \xi_\gamma = \xi_{\alpha+\beta} \cdot \xi_\gamma \]

\[
= 1/\sum_{n \geq 1} \left( \sum_{d|n} \mu_{\alpha+\beta}(d) \mu_\gamma \left( \frac{n}{d} \right) \right) /n^s = 1/\sum_{n \geq 1} \frac{\mu_{\alpha+\beta+\gamma}}{n^s} = \xi_{\alpha+\beta+\gamma}(s).
\]

Similarly, \(\xi_\alpha \cdot (\xi_\beta \cdot \xi_\gamma) = \xi_{\alpha+\beta+\gamma}\). Thus, \((\xi_\alpha \cdot \xi_\beta) \cdot \xi_\gamma = \xi_\alpha \cdot (\xi_\beta \cdot \xi_\gamma)\). It is also easy to check \(\xi_\alpha \cdot 1 = 1 \cdot \xi_\alpha = \xi_\alpha\) and \(\xi_\alpha \cdot \xi_{-\alpha} = \xi_{-\alpha} \cdot \xi_\alpha = 1\). Therefore, the theorem is proved. \(\square\)

From Theorem 2.1 and Equation (1.2) we have

**Corollary 2.1.** For all \(\alpha \in \mathbb{Z}\),

\[\xi_\alpha(s) = (\xi(s))^\alpha,\]

where \((\xi(s))^\alpha := \xi(s) (\xi(s))^{\alpha-1}\).

**Theorem 2.2.** Group \((M, \ast)\) and \((N, \cdot)\) are isomorphic.

**Proof.** Mapping \(\phi : M \mapsto N\) is defined by

\[\phi(\mu_\alpha) := \xi_\alpha = 1/\sum_{n \geq 1} \frac{\mu_\alpha(n)}{n^s},\]
where $\alpha \in \mathbb{C}$. It is easy to verify that the mapping is one-to-one and onto. In addition, for any $\alpha, \beta \in \mathbb{C}$,

$$\phi(\mu_\alpha * \mu_\beta) = \phi(\mu_{\alpha+\beta}) = \xi_{\alpha+\beta} = \xi_\alpha \cdot \xi_\beta = \phi(\mu_\alpha) \cdot \phi(\mu_\beta).$$

This completes the proof. \hfill \Box

3. SOME RESULTS FROM GENERALIZED ZETA FUNCTION GROUP

A series $\sum_{n \geq 1} a_n n^{-s}$ is called an arithmetic Dirichlet series if all of its coefficients $a_n = a(n)$ are arithmetic functions.

**Theorem 3.1.** (Generalized zeta inversion formulae) For any $\alpha \in \mathbb{C}$ and Dirichlet series $f$ and $g$,

$$f = \xi_\alpha g \iff g = \xi_{-\alpha} f.$$  

Moreover, if both $f = \sum_{n \geq 1} f_n n^{-s}$ and $g = \sum_{n \geq 1} g_n n^{-s}$ are arithmetic Dirichlet series, then for any $n \in \mathbb{N}$

$$f(n) = \sum_{d|n} \mu_\alpha \left( \frac{n}{d} \right) g(d) \iff g(n) = \sum_{d|n} \mu_{-\alpha} \left( \frac{n}{d} \right) f(d).$$

**Proposition 3.1.** For any $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}$,

$$\sum_{d|n} \mu_{\alpha-1}(d) = \mu_\alpha(n). \quad (3.1)$$

**Proof.** From Corollary 2.1

$$\sum_{n \geq 1} \frac{\mu_\alpha(n)}{n^s} = \frac{1}{\xi_\alpha(s)} = \frac{1}{\xi_{\alpha-1}(s)} \sum_{n \geq 1} \frac{\mu_{-1}(n)}{n^s} = \sum_{n \geq 1} \frac{\mu_{-1}(n)}{n^s} \cdot \sum_{n \geq 1} \frac{\mu_\alpha(n)}{n^s} = \sum_{n \geq 1} \sum_{d|n} \mu_{\alpha-1}(d) \mu_{-1}(\frac{n}{d}) / n^s,$$

which leads (3.1) by applying the Dirichlet series multiplication and noting that $\mu_{-1} \equiv 1$. \hfill \Box

**Proposition 3.2.** Let $f = \sum_{n \geq 1} c_n n^{-s}$, and let all $c_n$ be completely multiplicative functions. For any fixed positive integer $\alpha$,

$$f^{\alpha-1} \sum_{n \geq 1} \frac{\mu_\alpha(n) c_n}{n^{-s}} = \sum_{n \geq 1} \frac{\mu_\alpha(n) c_n}{n^{-s}}. \quad (3.2)$$
Proof. This follows easily from Proposition 3.1 and mathematical induction on $\alpha$. In fact, first we have
\[
f \sum_{n \geq 1}^{\infty} \frac{\mu_{\alpha}(n)c_n}{n^{-s}} = \sum_{n \geq 1}^{\infty} \left( (\mu_{\alpha}c) * c \right)(n) n^{-s} = \sum_{n \geq 1}^{\infty} \frac{c(n) \sum_{d|n} \mu_{\alpha}(d)}{n^{-s}} = \sum_{n \geq 1}^{\infty} \frac{c(n)\mu_{\alpha-1}(n)}{n^{-s}}.
\]
Secondly, using mathematical induction on $\alpha$ we obtain (3.2). □

It is known (cf., for examples, [4] and [6]) that for any fixed integer $\alpha \geq 1$
\[
\xi_{\alpha}^o(s) = \left( \sum_{n \geq 1}^{\infty} \mu(n)n^{-s} \right)^{\alpha} = \sum_{n \geq 1}^{\infty} r_{\alpha}(n)n^{-s}, \tag{3.3}
\]
where
\[
r_{\alpha}(n) = \sum_{n_1 \cdots n_{\alpha}=n} 1
\]
is the number of ways that $n$ can be written as a product of $\alpha$ fixed factors, so that $r_{\alpha}(n)$ is clearly a multiplicative function of $n$. In particular, $r_2(n)$ denotes the number of positive divisors of $n$.

**Theorem 3.2.** (Characteristic of generalized Möbius functions) For any fixed integer $\alpha \geq 1$, the inverse of $\mu_{\alpha}$ in the group $(M, \ast)$ is $r_{\alpha}$; or equivalently, $r_{\alpha} = \mu_{\alpha}$. Namely, for all integers $n \geq 1$
\[
(r_{\alpha} \ast \mu_{\alpha})(n) = \sum_{d|n} r_{\alpha}(d)\mu_{\alpha}(n/d) = \delta_{n,1}, \tag{3.4}
\]
where $\delta_{n,1} = 1$ if $n = 1$ and 0 otherwise.

Proof. Multiplying $\xi_{\alpha}^o(s)$ shown in (3.3) with
\[
\frac{1}{\xi_{\alpha}(s)} = \frac{1}{\xi_{\alpha}^o(s)} = \sum_{n \geq 1}^{\infty} \frac{\mu_{\alpha}(n)}{n^s}
\]
yields
\[
\sum_{n \geq 1}^{\infty} \frac{(r_{\alpha} \ast \mu_{\alpha})(n)}{n^s} = 1,
\]
which leads $r_{\alpha}(1)\mu_{\alpha}(1) = 1$ and
\[
(r_{\alpha} \ast \mu_{\alpha})(n) = \sum_{d|n} r_{\alpha}(d)\mu_{\alpha}(n/d) = 0
\]
for $n \geq 2$, completing the proof. □
Denote \( F_\alpha(s) = \sum_{n \geq 1} r_\alpha(n) n^{-s} \). Then we obtain
\[
F_\alpha(s) \xi_1 - \alpha(s) = \sum_{n \geq 1} \sum_{d|n} r_\alpha(d) \mu_{1-\alpha} \left( \frac{n}{d} \right) / n^s = \xi(s)
\]
and
\[
F_\alpha(s) \xi_{-\alpha}(s) = (\xi_1)^\alpha(\xi_1)^{-\alpha} = \xi_1^0 = \xi_0(s) = 1,
\]
i.e., identities (3.4) and \( \sum_{d|n} r_\alpha(d) \mu_{1-\alpha} \left( \frac{n}{d} \right) = 1 \). In particular, for \( r_2 = \sum_{d|n} 1 \), the number of positive divisors of \( n \), from \( \mu_{-2}(n) = r_2(n) \) we obtain
\[
F_2(s) \xi_{-1}(s) = 1 \quad \text{and} \quad F_2(s) \xi_{-2}(s) = \xi_0(s) = 1,
\]
i.e.,
\[
\sum_{d|n} r_2(d) \mu \left( \frac{n}{d} \right) = 1 \quad \text{and} \quad \sum_{d|n} r_2(d) \mu_2 \left( \frac{n}{d} \right) = \delta_{n,1}.
\]

ACKNOWLEDGMENTS

The author wish to thank the referee and editor for their helpful comments and suggestions.

REFERENCES