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ENHANCED $\left(\frac{G'}{G}\right)$-EXPANSION METHOD AND APPLICATIONS TO THE $(2+1)$D TYPICAL BREAKING SOLITON AND BURGERS EQUATIONS

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Dedicated to Professor E. M. E. Zayed
on the Occasion of his 60th Birthday

Abstract. In this paper, by introducing a new general framework, the enhanced $\left(\frac{G'}{G}\right)$-expansion method and its algorithm are proposed by studying Wang's $\left(\frac{G'}{G}\right)$-expansion method and constructing a first order nonlinear ordinary differential equation with a third-degree nonlinear term to the nonlinear $(2+1)$-dimensional typical breaking soliton and Burgers equations. As results, some new exact traveling wave solutions are obtained which include solitary wave solutions.

1. INTRODUCTION

In recent years, nonlinear wave equations have been played essential roles in many scientific and engineering areas such as fluid mechanics, plasma and elastic media and optical fibers, etc. Thus, it has had a considerable attention to find explicit traveling wave solutions of those problems. Several methods have been presented to obtain new exact solutions for many nonlinear evolution equations such that the homogeneous balance method [15], the tanh-function method [3], the algebraic method [4], the Hirota bilinear method [8], [9], the F-function expansion method [16], [17], the inverse scattering transform [1], the exp-function expansion method [7], the Jacobi elliptic function expansion [12], [19], the Bäcklund transform [13], [14], the sub-ODE method [11], the original $\left(\frac{G'}{G}\right)$-expansion method [18], the improved $\left(\frac{G'}{G}\right)$-expansion method [5], the generalized $\left(\frac{G'}{G}\right)$-expansion method [10], the extended $\left(\frac{G'}{G}\right)$-expansion method [6], the modified $\left(\frac{G'}{G}\right)$-expansion method [2] and so on. The objective of this article is to use the improved $\left(\frac{G'}{G}\right)$-expansion method which proposed by Guo et al [5] together with the generalized $\left(\frac{G'}{G}\right)$-expansion method which proposed by Lü et al [10], to find the exact solutions of nonlinear evolution equations

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via the (2+1)-dimensional typical breaking soliton and Burgers equations which play an important role in mathematical physics. The main idea of this method is that the traveling wave solutions of the nonlinear evolution equations can be expressed by polynomials in G where $G = G(\xi)$ is based on a first order nonlinear ordinary differential equation $G' = \sum_{i=0}^{3} h_i G^i$ with a third-degree nonlinear term, and $' = \frac{d}{d\xi}$. The degree of these polynomials can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in the given nonlinear equations. The coefficients of these polynomials can be obtained by solving a set of algebraic equations resulted from the process of using the proposed method. This method will play an important role in expressing the traveling wave solutions in terms of hyperbolic, trigonometric and rational functions for the nonlinear evolution equations in mathematical physics. The enhanced $(\frac{G'}{G})$-expansion method used in this article can be applied to further nonlinear equations as the difference-differential equations which can be done in forthcoming articles.

2. DESCRIPTION OF THE ENHANCED $(\frac{G'}{G})$-EXPANSION METHOD

Suppose that a nonlinear equation is given by

$$F(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, u_y, u_{yt}, u_{yy}, u_{xy}, ...) = 0, \tag{2.1}$$

where $u = u(x, y, t)$ is an unknown function, F is a polynomial in $u = u(x, y, t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. The main steps of the enhanced $(\frac{G'}{G})$-expansion method are the following:

Step 1. The traveling wave variable

$$u(x, y, t) = u(\xi), \quad \xi = x + y - Vt, \tag{2.2}$$

where V is a constant, permits us reducing Equation (2.1) into an ODE in the form

$$P(u, u', u'', ...) = 0, \tag{2.3}$$

where P is a polynomial in u and its total derivatives.

Step 2. Suppose that the solution of Equation (2.3) can be expressed by a polynomial in G as follows:

$$u(\xi) = \sum_{i=-m}^{m} \frac{A_i(\frac{G'}{G})^i}{\left[1 + \sigma(\frac{G'}{G})\right]^i}, \tag{2.4}$$

where $G = G(\xi)$ is the solution of the first order nonlinear ODE in the form

$$G' = h_0 + h_1 G + h_2 G^2 + h_3 G^3, \tag{2.5}$$

where σ, A_j $(-m, -m+1, ..., m-1, m)$, h_0, h_1, h_2 and h_3 are constants to be determined and $A_m \neq 0$, while m is called the balance number.

Step 3. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Equation (2.3) as follows: If we define the degree of $u(\xi)$ as $D[u(\xi)] = m$, then the degree of other expressions is defined by

$$D\left[u^r\left(\frac{d^q u}{d\xi^q}\right)^s\right] = mr + s\big(q+m\big). \tag{2.6}$$

Therefore, with (2.6), we can get the value of m in (2.4).

Step 4. Substituting (2.4) into (2.3) and using the ODE (2.5), collecting all terms with the same order of G together, we get a polynomial in G. Equating each coefficients of this polynomial to zero, yields a set of algebraic equations, which can be solved to get A_i and V. Since the general solution of Equation (2.5) is well known to us, then substituting A_i, V and the general solutions of (2.5) into (2.4), we have traveling wave solutions of Equation (2.1).

Remark 2.1. It is well known [10] that Equation (2.5) admits the following special solutions.

Theorem 2.1. *Suppose that* $h_0 = 0$, $h_2 = 0$ *and* $h_3 \neq 0$.

(i) *If* $h_1 \neq 0$, *then Equation* (2.5) *has the solutions*

$$G = \pm\frac{\sqrt{(-h_3 + C_1 h_1 \exp(-2h_1\xi)h_1}}{-h_3 + C_1 h_1 \exp(-2h_1\xi)}.$$

(ii) *If* $h_1 = 0$, *then Equation* (2.5) *has the solutions*

$$G = \pm\frac{1}{\sqrt{-2h_3\xi + C_2}}.$$

Theorem 2.2. *Suppose* $h_3 = 0$.

(i) *If* $h_0 \neq 0$, $h_1 \neq 0$ *and* $h_2 \neq 0$, *then Eq.* (2.5) *has the solutions*

$$G = -\frac{h_1}{2h_2} + \frac{\sqrt{4h_0h_2 - h_1^2}}{2h_2}\tan\left(\frac{\xi + C_3}{h_2}\sqrt{4h_0h_2 - h_1^2}\right).$$

(ii) *If* $h_0 \neq 0$, $h_1 \neq 0$ *and* $h_2 = 0$, *then Eq.* (2.5) *has the solution*

$$G = \frac{h_0}{-h_1} + C_4\exp(h_1\xi).$$

(iii) *If* $h_0 = 0$, $h_1 \neq 0$ *and* $h_2 \neq 0$, *then Eq.* (2.5) *has the solution*

$$G = \frac{h_1}{-h_2 + C_5 h_1 \exp(-h_1\xi)}.$$

(iv) *If* $h_0 = 0$, $h_1 = 0$ *and* $h_2 \neq 0$, *then Eq.* (2.5) *has the solution*

$$G = \frac{1}{-h_2\xi + C_6}.$$

(v) *If* $h_0 \neq 0,\ h_1 = 0,$ *and* $h_2 = 0,$ *then Eq.* (2.5) *has the solution*

$$G = h_0\xi + C_7,$$

where $C_i\ (0,1,...,7)$ *are arbitrary constants.*

3. APPLICATIONS

In this section, we apply the enhanced $\left(\frac{G'}{G}\right)$-expansion method to find new traveling wave solutions for some nonlinear PDEs in mathematical physics.

3.1. **The** $(2+1)$**-dimensional typical breaking soliton equation.** Consider the following $(2+1)$-dimensional typical breaking soliton equation

$$u_{xt} - 4u_x u_{xy} - 2u_{xx}u_y + u_{xxxy} = 0. \tag{3.1}$$

The traveling wave variable (2.2) permits us converting Eq. (3.1) into the following ODE:

$$C - Vu' - 3\big(u'\big)^2 + u''' = 0, \tag{3.2}$$

where C is an integration constant. Consider the homogeneous balance between u''' and $(u')^2$ in (3.2) we get $m = 1$. From (2.4) we get

$$u(\xi) = \sum_{i=-1}^{1} \frac{A_i\left(\frac{G'}{G}\right)^i}{\left[1+\sigma\left(\frac{G'}{G}\right)\right]^i}, \tag{3.3}$$

where A_i are constants to be determined and $G = G(\xi)$ satisfies Eq. (2.5). It is easy to deduce that:

$$\begin{aligned} u' = {} & \frac{h_0 - h_2G^2 - 2h_3G^3}{(h_0 + h_1G + h_2G^2 + h_3G^3)(\sigma h_3G^3 + \sigma h_2G^2 + (1+\sigma h_1)G + \sigma h_0)^2} \\ & \times[(A_2\sigma^2h_3^2 - A_1h_3^2)G^6 + (2A_2\sigma^2h_2h_3 - 2A_1h_2h_3)G^5 \\ & +(A_2\sigma^2h_2^2 + 2A_2\sigma h_3 - A_1h_2^2 + 2A_2\sigma^2h_1h_3 - 2A_1h_1h_3)G^4 \\ & +(-2A_1h_1h_2 - 2A_1h_0h_3 + 2A_2\sigma^2h_3h_0 + 2A_2\sigma h_2 + 2A_2\sigma^2h_1h_2)G^3 \\ & +(A_2 + 2A_2\sigma h_1 - A_1h_1^2 + A_2\sigma^2h_1^2 - 2A_1h_0h_2 + 2A_2\sigma^2h_2h_0)G^2 \\ & +(-2A_1h_0h_1 + 2A_2\sigma h_0 + 2A_2\sigma^2h_1h_0)G - A_1h_0^2 + A_2\sigma^2h_0^2]; \end{aligned} \tag{3.4}$$

$$\begin{aligned} u'' = {} & \frac{-1}{(h_0 + h_1G + h_2G^2 + h_3G^3)(\sigma h_3G^3 + \sigma h_2G^2 + (1+\sigma h_1)G + \sigma h_0)^3} \\ & \times[(-A_1\sigma h_3^4h_2 + A_2\sigma^3h_3^4h_2)G^{13} + (-8A_1h_3^4 - 4A_1\sigma h_3^4h_1 - 3A_1\sigma h_3^3h_2^2 \\ & +4A_2\sigma^3h_3^4h_1 + 3A_2\sigma^3h_3^3h_2^2)G^{12} + (-25A_1h_3^3h_2 - 3A_1\sigma h_3^2h_2^3 + 3A_2\sigma^2h_3^3h_2 \\ & -16A_1\sigma h_3^3h_1h_2 - 9A_1\sigma h_3^4h_0 + 3A_2\sigma^3h_3^2h_2^3 + 9A_2\sigma^3h_3^4h_0 + 16A_2\sigma^3h_3^3h_1h_2)G^{11} \\ & +(12A_2\sigma^3h_3^3h_1^2 - 21A_1\sigma h_3^2h_1h_2^2 - 12A_1\sigma h_3^3h_1^2 - 28A_1h_3^2h_2^2 - 34A_1\sigma h_3^3h_0h_2 \end{aligned}$$

$$+21A_2\sigma^3h_3^2h_1h_2^2 - A_1\sigma h_2^4h_3 + 12A_2\sigma^2h_3^3h_1 + 6A_2\sigma^2h_3^2h_2^2 + A_2\sigma^3h_3h_2^4$$
$$-20A_1h_3^3h_1 + 34A_2\sigma^3h_3^3h_0h_2)G^{10} + (45A_2\sigma^3h_3^2h_2^2h_0 + 3A_2\sigma h_3^2h_2 - 17A_1h_0h_3^3$$
$$-13A_1h_3h_2^3 - 40A_1\sigma h_3^3h_0h_1 + 33A_2\sigma^2h_3^2h_1h_2 + 40A_2\sigma^3h_3^3h_0h_1 + 30A_2\sigma^3h_3^2h_1^2h_2$$
$$+10A_2\sigma^3h_3h_1h_2^3 - 30A_1\sigma h_3^2h_1^2h_2 - 10A_1\sigma h_1h_3h_2^3 - 45A_1\sigma h_3^2h_0h_2^2 - 43A_1h_3^2h_1h_2$$
$$+27h_3^3A_2\sigma^2h_0 + 3A_2\sigma^2h_3h_2^3)G^9 + (-28A_1h_3h_1h_2^2 - A_1\sigma h_2^4h_1 - 27A_1\sigma h_3^3h_0^2$$
$$+24h_2^3A_2\sigma^3h_3h_0 + 102A_2\sigma^3h_3^2h_1h_0h_2 - 21A_1\sigma h_1^2h_3h_2^2 + 12A_2\sigma h_3^2h_1 - 2h_2^4A_1$$
$$+3A_2\sigma h_3h_2^2 + 24A_2\sigma^2h_3h_1h_2^2 - 102A_1\sigma h_3^2h_0h_1h_2 - 16A_1h_3^2h_1^2 + 21A_2\sigma^3h_3h_1^2h_2^2$$
$$+27A_2\sigma^3h_3^3h_0^2 + 24A_2\sigma^2h_3^2h_1^2 + 72A_2\sigma^2h_3^2h_0h_2 - 24h_2^3A_1h_3\sigma h_0 + 12h_3^2A_2\sigma^3h_1^3$$
$$-36A_1h_0h_2h_3^2 + A_2\sigma^3h_2^4h_1 - 12h_3^2A_1h_1^3\sigma)G^8 + (78A_2\sigma^3h_3h_1h_2^2h_0 + 3A_2\sigma^3h_2^3h_1^2$$
$$-16h_2A_1h_1^3\sigma h_3 + 4h_2^4A_2\sigma^3h_0 - 4h_2^4A_1\sigma h_0 + 27A_2\sigma h_0h_3^2 - 23A_1h_0h_2^2h_3$$
$$+81A_2\sigma^2h_3^2h_0h_1 + A_2h_2h_3 + 3A_2\sigma^2h_2^3h_1 - 54A_1\sigma h_3^2h_0h_1^2 - 19A_1h_3h_1^2h_2$$
$$-3A_1\sigma h_2^3h_1^2 + 69A_2\sigma^3h_0^2h_2h_3^2 - 78A_1\sigma h_1h_3h_0h_2^2 - 27A_1h_0h_1h_3^2 + 18A_2\sigma h_3h_1h_2$$
$$+54A_2\sigma^3h_3^2h_1^2h_0 - 69A_1\sigma h_3^2h_0^2h_2 + 16h_2A_2\sigma^3h_1^3h_3 + 33A_2\sigma^2h_3h_1^2h_2 - 5h_2^3A_1h_1$$
$$+57A_2\sigma^2h_3h_2^2h_0)G^7 + (-12A_1h_0^2h_3^2 - 4A_1h_0h_2^3 - 4A_1\sigma h_1^4h_3 - 3A_1\sigma h_2^2h_1^3$$
$$+54A_2\sigma^2h_3^2h_0^2 + 12A_2\sigma h_3h_1^2 + 12A_2\sigma^2h_3h_1^3 + 4A_2\sigma^3h_3h_1^4 - 4A_1h_3h_1^3$$
$$-32A_1h_0h_1h_2h_3 + 4A_2h_3h_1 - 4h_2^2A_1h_1^2 - 54A_1\sigma h_2^2h_3h_0^2 - 69A_1\sigma h_3^2h_0^2h_1$$
$$-78A_1\sigma h_1^2h_3h_0h_2 + 69A_2\sigma^3h_0^2h_1h_3^2 + 54A_2\sigma^3h_0^2h_2^2h_3 - 16A_1\sigma h_1h_0h_2^3$$
$$+42A_2\sigma h_0h_2h_3 + 12A_2\sigma^2h_0h_2^3 + 3A_2\sigma h_2^2h_1 + 6A_2\sigma^2h_2^2h_1^2 + 3A_2\sigma^3h_2^2h_1^3$$
$$+120A_2\sigma^2h_3h_1h_0h_2 + 78A_2\sigma^3h_3h_1^2h_0h_2 + 16A_2\sigma^3h_0h_2^3h_1)G^6$$
$$+(-15A_1h_0^2h_2h_3 - 11A_1h_0h_1^2h_3 - 7A_1h_0h_1h_2^2 - 27A_1\sigma h_3^2h_0^3 - 12A_1\sigma h_2^3h_0^2$$
$$+27A_2\sigma^3h_0^3h_3^2 + 12A_2\sigma^3h_0^2h_2^3 + 12A_2\sigma h_0h_2^2 + 9A_2h_0h_3 - h_2A_1h_1^3 + h_2A_2h_1$$
$$-24A_1\sigma h_1^3h_3h_0 - 102A_1\sigma h_1h_3h_0^2h_2 + 102A_2\sigma^3h_0^2h_1h_2h_3 - 21A_1\sigma h_1^2h_0h_2^2$$
$$+42A_2\sigma h_0h_1h_3 - h_2A_1h_1^4\sigma + 3h_2A_2h_1^2\sigma + 3h_2A_2\sigma^2h_1^3 + h_2A_2\sigma^3h_1^4$$
$$+57A_2\sigma^2h_3h_1^2h_0 + 81A_2\sigma^2h_3h_2h_0^2 + 24A_2\sigma^3h_3h_1^3h_0 + 33A_2\sigma^2h_0h_1h_2^2$$
$$+21A_2\sigma^3h_0h_1^2h_2^2)G^5 + (24A_2\sigma^2h_0h_1^2h_2 + 4A_2h_0h_2 + 10A_2\sigma^3h_0h_1^3h_2$$
$$-40A_1\sigma h_2h_3h_0^3 + 27A_2\sigma h_0^2h_3 + 18A_2\sigma h_0h_1h_2 + 24A_2\sigma^2h_0^2h_2^2 + 30A_2\sigma^3h_0^2h_1h_2^2$$
$$-4A_1h_0h_1^2h_2 - 30A_1\sigma h_1h_0^2h_2^2 + 72A_2\sigma^2h_3h_1h_0^2 + 40A_2\sigma^3h_0^3h_2h_3 - 45A_1\sigma h_1^2h_3h_0^2$$

$$-4A_1h_0^2h_2^2+45A_2\sigma^3h_0^2h_1^2h_3-12A_1h_0^2h_1h_3-10A_1\sigma h_1^3h_0h_2)G^4$$
$$+(-12A_1\sigma h_2^2h_0^3+A_2\sigma^3h_0h_1^4-5A_1h_0^3h_3+33A_2\sigma^2h_0^2h_1h_2+21A_2\sigma^3h_0^2h_1^2h_2$$
$$-A_1\sigma h_1^4h_0-21A_1\sigma h_1^2h_0^2h_2+12A_2\sigma h_0^2h_2+3A_2\sigma h_0h_1^2-34A_1\sigma h_1h_3h_0^3$$
$$+34A_2\sigma^3h_0^3h_1h_3+27A_2\sigma^2h_0^3h_3+3A_2\sigma^2h_0h_1^3-7A_1h_0^2h_1h_2+A_2h_0h_1$$
$$+12A_2\sigma^3h_0^3h_2^2-A_1h_0h_1^3)G^3+(3A_2\sigma h_0^2h_1-4A_1h_0^3h_2-9A_1\sigma h_3h_0^4+9A_2\sigma^3h_0^4h_3$$
$$+12A_2\sigma^2h_0^3h_2-16A_1\sigma h_1h_0^3h_2+6A_2\sigma^2h_0^2h_1^2-3A_1\sigma h_1^3h_0^2-4A_1h_0^2h_1^2+3A_2\sigma^3h_0^2h_1^3$$
$$+16A_2\sigma^3h_0^3h_1h_2)G^2+(3A_2\sigma^2h_0^3h_1-5A_1h_0^3h_1+3A_2\sigma^3h_0^3h_1^2+4A_2\sigma^3h_0^4h_2$$
$$-4A_1\sigma h_2h_0^4-3A_1\sigma h_1^2h_0^3)G-2A_1h_0^4+A_2\sigma^3h_0^4h_1-A_1\sigma h_1h_0^4]; \qquad (3.5)$$

$$u'''=\frac{-1}{(h_0+h_1G+h_2G^2+h_3G^3)(\sigma h_3G^3+\sigma h_2G^2+(1+\sigma h_1)G+\sigma h_0)^4}$$
$$\times[(A_2\sigma^4h_3^6h_2-A_1\sigma^2h_3^6h_2)G^{18}+(6A_2\sigma^4h_3^5h_2^2-6A_1\sigma^2h_3^5h_2^2)G^{17}$$
$$+(14A_2\sigma^4h_2^3h_3^4-14A_1\sigma h_2h_3^5+10A_2\sigma^4h_3^5h_1h_2-14A_1\sigma^2h_3^4h_2^3$$
$$-9A_2\sigma^4h_0h_3^6+9A_1h_0\sigma^2h_3^6-10A_1\sigma^2h_3^5h_1h_2+4A_2\sigma^3h_3^5h_2)G^{16}$$
$$+(-8A_1\sigma^2h_3^5h_1^2-44A_1\sigma^2h_3^4h_2^2h_1-16A_1\sigma^2h_3^3h_2^4-48A_1\sigma h_3^5h_1+44A_2\sigma^4h_3^4h_2^2h_1$$
$$+8A_2\sigma^4h_3^5h_1^2+20A_2\sigma^3h_3^4h_2^2+16A_2\sigma^4h_2^4h_3^3+36A_1h_0\sigma^2h_2h_3^5-36A_2\sigma^4h_0h_3^5h_2$$
$$-48A_1h_3^5-52A_1\sigma h_2^2h_3^4)G^{15}+(-72A_1\sigma^2h_3^3h_2^3h_1-181A_1h_3^4h_2+63A_2\sigma^4h_3^4h_2h_1^2$$
$$-90A_1h_0\sigma h_3^5-9A_1\sigma^2h_3^2h_2^5+9A_2\sigma^4h_2^5h_3^2-18A_2\sigma^4h_0h_3^5h_1+72A_2\sigma^4h_2^3h_3^3h_1$$
$$+6A_2\sigma^2h_3^4h_2-72A_1\sigma h_2^3h_3^3+54A_1h_0\sigma^2h_2^2h_3^4+36A_2\sigma^3h_3^3h_2^3-54A_2\sigma^4h_0h_3^4h_2^2$$
$$+18A_1h_0\sigma^2h_1h_3^5-63A_1\sigma^2h_3^4h_1^2h_2-36A_2\sigma^3h_0h_3^5-234A_1\sigma h_2h_3^4h_1$$
$$+36A_2\sigma^3h_3^4h_1h_2)G^{14}+(-262A_1h_3^3h_2^2-144A_1h_3^4h_1+32h_1^2A_2\sigma^3h_3^4$$
$$-18A_2\sigma^4h_0^2h_3^5+24A_2\sigma^2h_3^3h_2^2+28A_2\sigma^3h_3^2h_2^4+32A_2\sigma^4h_1^3h_3^4+2A_2\sigma^4h_2^6h_3$$
$$-16A_2\sigma^4h_0h_3^4h_1h_2+40A_1h_0\sigma^2h_2^3h_3^3+16A_1h_0\sigma^2h_3^4h_1h_2-400A_1h_0\sigma h_2h_3^4$$
$$-408A_1\sigma h_2^2h_3^3h_1-148A_1\sigma^2h_3^3h_1^2h_2^2-52A_1\sigma^2h_3^2h_2^4h_1-40A_2\sigma^4h_0h_2^3h_3^3$$
$$+148A_2\sigma^4h_3^3h_2^2h_1^2+120A_2\sigma^3h_3^3h_2^2h_1+52A_2\sigma^4h_2^4h_3^2h_1-112A_2\sigma^3h_0h_3^4h_2$$
$$+18A_1h_0^2\sigma^2h_3^5-160A_1\sigma h_3^4h_1^2-44A_1\sigma h_2^4h_3^2-32A_1\sigma^2h_3^4h_1^3-2A_1\sigma^2h_3h_2^6)G^{13}$$
$$+(-404A_1h_3^3h_1h_2+184A_2\sigma^3h_3^3h_1^2h_2+49A_2\sigma^4h_0h_3^4h_1^2+4A_2\sigma h_2h_3^3+30A_2\sigma^2h_2^3h_3^2$$

$$+8A_2\sigma^3h_2^5h_3-123A_1h_0h_3^4-42A_2\sigma^4h_0^2h_3^4h_2+56A_2\sigma^4h_0h_1h_2^2h_3^3-179A_1h_3^2h_2^3$$
$$-49A_1h_0\sigma^2h_1^2h_3^4+21A_1h_0\sigma^2h_2^4h_3^2-56A_1h_0\sigma^2h_2^2h_3^3h_1+42A_1h_0^2\sigma^2h_3^4h_2$$
$$-438A_1h_0\sigma h_1h_3^4-664A_1h_0\sigma h_2^2h_3^3-14A_1\sigma^2h_3h_2^5h_1-54A_2\sigma^2h_0h_3^4$$
$$-312A_1\sigma h_2^3h_3^2h_1-524A_1\sigma h_2h_3^3h_1^2-140A_1\sigma^2h_3^3h_1^3h_2-140A_1\sigma^2h_3^2h_2^3h_1^2$$
$$-21A_2\sigma^4h_0h_2^4h_3^2+48A_2\sigma^2h_3^3h_1h_2+140A_2\sigma^4h_2^3h_3^2h_1^2+140A_2\sigma^4h_1^3h_2h_3^3$$
$$+132A_2\sigma^3h_3^2h_2^3h_1+14A_2\sigma^4h_2^5h_3h_1-36A_2\sigma^3h_0h_3^4h_1-124A_2\sigma^3h_0h_3^3h_2^2$$
$$-10A_1\sigma h_2^5h_3)G^{12}+(-56A_1h_2^4h_3-396A_1h_3^2h_1h_2^2-36A_2\sigma^3h_0^2h_3^4+48A_2\sigma^2h_3^3h_1^2$$
$$+12A_2\sigma h_2^2h_3^2+12A_2\sigma^2h_2^4h_3+96A_2\sigma^3h_1^3h_3^3+48A_2\sigma^4h_1^4h_3^3+48A_2\sigma^3h_0h_3^3h_1h_2$$
$$-152A_1h_3^3h_1^2+36A_2\sigma^4h_0^2h_3^4h_1+264A_2\sigma^4h_0h_3^3h_1^2h_2-24A_2\sigma^4h_0^2h_2^2h_3^3$$
$$+72A_2\sigma^4h_0h_1h_2^3h_3^2-120A_2\sigma^2h_0h_2h_3^3+12A_1h_0\sigma^2h_2^5h_3-264A_1h_0\sigma^2h_1^2h_3^3h_2$$
$$-72A_1h_0\sigma^2h_2^3h_3^2h_1+24A_1h_0^2\sigma^2h_2^2h_3^3-36A_1h_0^2\sigma^2h_3^4h_1-504A_1h_0\sigma h_2^3h_3^2$$
$$-192A_1\sigma^2h_3^2h_1^3h_2^2-48A_1\sigma^2h_3h_2^4h_1^2-1392A_1h_0\sigma h_3^3h_1h_2-576A_1\sigma h_2^2h_3^2h_1^2$$
$$-96A_1\sigma h_2^4h_3h_1-12A_2\sigma^4h_0h_2^5h_3+288A_2\sigma^3h_3^2h_2^2h_1^2+48A_2\sigma^4h_2^4h_3h_1^2$$
$$+192A_2\sigma^4h_1^3h_2^2h_3^2+108A_2\sigma^2h_1h_3^2h_2^2+48A_2\sigma^3h_2^4h_3h_1-72A_2\sigma^3h_0h_3^2h_2^3$$
$$-340A_1h_0h_2h_3^3-252A_1h_0^2\sigma h_3^4-192A_1\sigma h_3^3h_1^3-48A_1\sigma^2h_3^3h_1^4)G^{11}$$
$$+(A_2h_2h_3^2-154A_1h_1h_3h_2^3-282A_1h_3^2h_1^2h_2-6h_1A_1\sigma h_2^5+28A_2h_3^2h_2\sigma h_1$$
$$+8A_2\sigma h_2^3h_3+A_2\sigma^4h_2^5h_1^2+200A_2\sigma^3h_0h_3^3h_1^2+372A_2\sigma^4h_0h_1^2h_2^2h_3^2$$
$$-20A_2\sigma^3h_0^2h_3^3h_2-6h_2^5A_1+2A_2\sigma^4h_0h_2^4h_1h_3+232A_2\sigma^4h_0^2h_3^3h_1h_2$$
$$+180A_2\sigma^3h_0h_2^2h_3^2h_1-6A_2\sigma^4h_0^2h_2^3h_3^2-90A_2\sigma^2h_0h_2^2h_3^2-164A_1h_0\sigma^2h_1^3h_3^3$$
$$-232A_1h_0^2\sigma^2h_3^3h_1h_2-372A_1h_0\sigma^2h_1^2h_2^2h_3^2+6A_1h_0^2\sigma^2h_2^3h_3^2-2A_1h_0\sigma^2h_1h_2^4h_3$$
$$-676A_1h_0\sigma h_1^2h_3^3-166A_1h_0\sigma h_2^4h_3-88A_1\sigma^2h_3h_2^3h_1^3-36A_2\sigma h_0h_3^3+18A_2\sigma^4h_0^3h_3^4$$
$$-1512A_1h_0\sigma h_2^2h_1h_3^2-776A_1h_0^2\sigma h_3^3h_2-404A_1\sigma h_2h_3^2h_1^3-232A_1\sigma h_2^3h_3h_1^2$$
$$-127A_1\sigma^2h_3^2h_1^4h_2-40A_2\sigma^3h_0h_2^4h_3+164A_2\sigma^4h_0h_1^3h_3^3+180A_2\sigma^2h_2h_3^2h_1^2$$
$$+140A_2\sigma^3h_2^3h_3h_1^2+88A_2\sigma^4h_1^3h_2^3h_3+60A_2\sigma^2h_2^3h_3h_1+280A_2\sigma^3h_1^3h_3^2h_2$$
$$+127A_2\sigma^4h_1^4h_3^2h_2+4A_1h_0\sigma^2h_2^6-18A_1h_0^3\sigma^2h_3^4-252A_1h_0h_1h_3^3$$
$$-327A_1h_0h_2^2h_3^2-A_1\sigma^2h_2^5h_1^2-4A_2\sigma^4h_0h_2^6)G^{10}+(-64h_3^2A_1h_1^3-18h_2^4A_1h_1$$
$$-102A_1h_0^2h_3^3+2A_2h_2^2h_3+200A_2\sigma^3h_1^3h_2^2h_3+86A_2\sigma^4h_1^4h_2^2h_3-150A_1h_1^2h_3h_2^2$$

$$-20A_1\sigma h_2^4h_1^2+144A_2\sigma^2h_3h_1^2h_2^2+32A_2\sigma h_3^2h_1^2+96A_2\sigma^2h_1^3h_3^2+96A_2\sigma^3h_1^4h_3^2$$
$$+32A_2\sigma^4h_1^5h_3^2+4A_2\sigma^4h_2^4h_1^3+4A_2\sigma^3h_2^4h_1^2-48A_2\sigma h_0h_2h_3^2+336A_2\sigma^4h_0^2h_2^2h_3^2h_1$$
$$+432A_2\sigma^4h_0h_1^3h_2h_3^2+216A_2\sigma^3h_0^2h_3^3h_1+624A_2\sigma^3h_0h_3^2h_1^2h_2-18A_2\sigma^4h_0^2h_2^4h_3$$
$$+136A_2\sigma^4h_0h_2^3h_1^2h_3+104A_2\sigma^4h_0^3h_3^3h_2+48A_2\sigma^3h_0^2h_2^2h_3^2+260A_2\sigma^4h_0^2h_3^3h_1^2$$
$$-48A_2\sigma^2h_0h_2^3h_3-456A_1h_0h_1h_2h_3^2+16A_1h_0\sigma^2h_1h_2^5-336A_1h_0^2\sigma^2h_2^2h_1h_3^2$$
$$-104A_1h_0^3\sigma^2h_3^3h_2-136A_1h_0\sigma^2h_1^2h_2^3h_3-432A_1h_0\sigma^2h_1^3h_3^2h_2-96A_1\sigma h_3^2h_1^4$$
$$-260A_1h_0^2\sigma^2h_1^2h_3^3+18A_1h_0^2\sigma^2h_2^4h_3-624A_1h_0\sigma h_2^3h_1h_3-1392A_1h_0\sigma h_1^2h_2h_3^2$$
$$-86A_1\sigma^2h_3h_2^2h_1^4-16A_2\sigma^3h_0h_2^5-816A_1h_0^2\sigma h_2^2h_3^2-696A_1h_0^2\sigma h_3^3h_1$$
$$-232A_1\sigma h_2^2h_3h_1^3-4A_1\sigma^2h_2^4h_1^3-16A_2\sigma^4h_0h_2^5h_1+32A_2\sigma h_2^2h_3h_1-16A_1h_0\sigma h_2^5$$
$$+144A_2\sigma^2h_0h_3^2h_1h_2+48A_2\sigma^3h_0h_2^3h_1h_3-124A_1h_0h_2^3h_3-32A_1\sigma^2h_3^2h_1^5)G^9$$
$$+(-19h_2^3A_1h_1^2-14A_1h_0h_2^4-9A_2h_0h_3^2+132A_2\sigma^3h_1^4h_2h_3+42A_2\sigma^4h_1^5h_2h_3$$
$$+6A_2h_2h_3h_1-60h_2A_1h_1^3h_3-24A_1\sigma h_2^3h_1^3+252A_2\sigma^2h_0h_3^2h_1^2-21A_2\sigma^4h_0h_2^4h_1^2$$
$$-12A_2\sigma^4h_0^2h_2^5+12A_2\sigma^3h_2^3h_1^3+6A_2\sigma^2h_1^2h_2^3+6A_2\sigma^4h_1^4h_2^3-24A_2\sigma h_0h_2^2h_3$$
$$+36A_2\sigma h_0h_1h_3^2+684A_2\sigma^4h_0^2h_1^2h_3^2h_2+150A_2\sigma^4h_0^3h_2^2h_3^2+264A_2\sigma^4h_0h_1^3h_2^2h_3$$
$$+684A_2\sigma^3h_0^2h_3^2h_1h_2+108A_2\sigma^4h_0^2h_2^3h_1h_3+216A_2\sigma^4h_0^3h_3^3h_1-48A_2\sigma^3h_0h_2^4h_1$$
$$-234A_1h_0h_1h_2^2h_3+21A_1h_0\sigma^2h_1^2h_2^4-153A_1h_0\sigma^2h_1^4h_3^2-108A_1h_0^2\sigma^2h_2^3h_1h_3$$
$$-150A_1h_0^3\sigma^2h_2^2h_3^2-216A_1h_0^3\sigma^2h_3^3h_1-684A_1h_0^2\sigma^2h_1^2h_2h_3^2-264A_1h_0\sigma^2h_1^3h_2^2h_3$$
$$-1368A_1h_0^2\sigma h_1h_2h_3^2-396A_1h_0\sigma h_1^3h_3^2-66A_1h_0\sigma h_1h_2^4-792A_1h_0\sigma h_1^2h_2^2h_3$$
$$-42A_1\sigma^2h_3h_1^5h_2+108A_2\sigma^3h_0^3h_3^3-324A_1h_0^2\sigma h_2^3h_3-102A_1\sigma h_2h_3h_1^4$$
$$+60A_2\sigma h_2h_3h_1^2+144A_2\sigma^2h_1^3h_2h_3+360A_2\sigma^3h_0h_1^3h_3^2+153A_2\sigma^4h_0h_1^4h_3^2$$
$$+108A_2\sigma^2h_0h_2^2h_1h_3+90A_2\sigma^2h_0^2h_3^2h_2+396A_2\sigma^3h_0h_2^2h_1^2h_3-177A_1h_0^2h_2h_3^2$$
$$-150A_1h_0h_1^2h_3^2+12A_1h_0^2\sigma^2h_2^5-216A_1h_0^3\sigma h_3^3-6A_1\sigma^2h_2^3h_1^4-24A_2\sigma^2h_0h_2^4)G^8$$
$$+(-8h_2^2A_1h_1^3+8A_2h_3h_1^2-8A_1h_1^4h_3-16A_1\sigma h_1^5h_3-12A_1\sigma h_2^2h_1^4+32A_2h_3h_1^3\sigma$$
$$+4A_2h_2^2h_1^2\sigma-8A_1\sigma^2h_1^6h_3-4A_1\sigma^2h_2^2h_1^5+48A_2\sigma^2h_3h_1^4+32A_2\sigma^3h_3h_1^5$$
$$+12A_2\sigma^2h_2^2h_1^3+12A_2\sigma^3h_2^2h_1^4+8A_2\sigma^4h_3h_1^6+4A_2\sigma^4h_2^2h_1^5-40A_2\sigma^3h_0h_2^3h_1^2$$
$$-32A_2\sigma^3h_0^2h_2^4-4A_2h_0h_2h_3+80A_2\sigma h_0h_1h_2h_3+408A_2\sigma^4h_0^2h_1^2h_2^2h_3$$
$$+568A_2\sigma^4h_0^3h_1h_3^2h_2+336A_2\sigma^4h_0^2h_1^3h_3^2+328A_2\sigma^3h_0^3h_3^2h_2+360A_2\sigma^2h_0h_1^2h_2h_3$$

$$-32A_2\sigma^4h_0^2h_2^4h_1+56A_2\sigma^4h_0^3h_2^3h_3+624A_2\sigma^3h_0^2h_3^2h_1^2-48A_2\sigma^2h_0h_1h_2^3$$
$$-132A_1h_0h_1^2h_2h_3+8A_1h_0\sigma^2h_1^3h_2^3-568A_1h_0^3\sigma^2h_1h_2h_3^2-56A_1h_0^3\sigma^2h_2^3h_3$$
$$+32A_1h_0^2\sigma^2h_1h_2^4-408A_1h_0^2\sigma^2h_1^2h_2^2h_3-336A_1h_0^2\sigma^2h_1^3h_3^2-720A_1h_0^2\sigma h_1h_2^2h_3$$
$$-188A_1h_0\sigma^2h_1^4h_2h_3-88A_1h_0\sigma h_1^2h_2^3-400A_1h_0\sigma h_1^3h_2h_3-528A_1h_0^2\sigma h_1^2h_3^2$$
$$+36A_2\sigma h_0^2h_3^2-16A_2\sigma h_0h_2^3+72A_2\sigma^4h_0^4h_3^3-392A_1h_0^3\sigma h_3^2h_2-8A_2\sigma^4h_0h_1^3h_2^3$$
$$+72A_2\sigma^2h_0^2h_2^2h_3+324A_2\sigma^2h_0^2h_3^2h_1+464A_2\sigma^3h_0h_1^3h_2h_3+432A_2\sigma^3h_0^2h_2^2h_1h_3$$
$$+188A_2\sigma^4h_0h_1^4h_2h_3-108A_1h_0^2h_1h_3^2-84A_1h_0^2h_2^2h_3-72A_1h_0^4\sigma^2h_3^3-32A_1h_0^2\sigma h_2^4$$
$$-28A_1h_0h_1h_2^3)G^7+(-h_2A_1h_1^4-8A_1h_0^2h_2^3-4A_2h_0h_2^2+156A_2\sigma^3h_0h_1^4h_3$$
$$+46A_2\sigma^4h_0h_1^5h_3-24A_2\sigma^2h_0^2h_2^3+18A_2h_0h_3h_1-16A_2\sigma h_0h_1h_2^2$$
$$+72A_2\sigma h_0^2h_2h_3+356A_2\sigma^4h_0^3h_1h_2^2h_3+408A_2\sigma^4h_0^2h_1^3h_2h_3+404A_2\sigma^4h_0^3h_1^2h_3^2$$
$$+189A_2\sigma^4h_0^4h_2h_3^2+540A_2\sigma^3h_0^3h_3^2h_1+468A_2\sigma^2h_0^2h_1h_2h_3+192A_2\sigma^2h_0h_1^3h_3$$
$$-18A_2\sigma^2h_0h_2^2h_1^2-22A_2\sigma^4h_0^2h_2^3h_1^2+h_2A_2h_1^2-46A_1h_0\sigma^2h_1^5h_3-2A_1h_0\sigma^2h_1^4h_2^2$$
$$-356A_1h_0^3\sigma^2h_1h_2^2h_3-189A_1h_0^4\sigma^2h_3^2h_2-78A_1h_0^2h_1h_2h_3-408A_1h_0^2\sigma^2h_1^3h_2h_3$$
$$-404A_1h_0^3\sigma^2h_1^2h_3^2+22A_1h_0^2\sigma^2h_1^2h_2^3-66A_1h_0\sigma h_1^4h_3-40A_1h_0\sigma h_1^3h_2^2$$
$$-172A_1h_0^3\sigma h_2^2h_3+4h_2A_2\sigma^3h_1^5+100A_2\sigma h_0h_1^2h_3+162A_2\sigma^2h_0^3h_3^2-8A_2\sigma^4h_0^3h_2^4$$
$$-456A_1h_0^2\sigma h_2h_1^2h_3-60A_1h_0^2\sigma h_2^3h_1-264A_1h_0^3\sigma h_3^2h_1-4A_2\sigma^3h_0h_1^3h_2^2$$
$$+2A_2\sigma^4h_0h_1^4h_2^2+804A_2\sigma^3h_0^2h_1^2h_2h_3-h_2A_1h_1^6\sigma^2-48A_2\sigma^3h_0^2h_2^3h_1$$
$$+224A_2\sigma^3h_0^3h_2^2h_3-21A_1h_0^3h_3^2+6h_2A_2\sigma^2h_1^4-20A_1h_0h_1^3h_3+4h_2A_2h_1^3\sigma$$
$$-2h_2A_1h_1^5\sigma+h_2A_2\sigma^4h_1^6-15A_1h_0h_1^2h_2^2+8A_1h_0^3\sigma^2h_2^4)G^6+(18A_2h_0^2h_3$$
$$-36A_1h_0^4\sigma h_3^2+252A_2\sigma^4h_0^4h_1h_3^2+180A_2\sigma^3h_0^4h_3^2+504A_2\sigma^4h_0^3h_1^2h_2h_3$$
$$+360A_2\sigma^3h_0^2h_1^3h_3+360A_2\sigma^2h_0^2h_1^2h_3-4A_1h_0^3h_2h_3-72A_1h_0^2\sigma h_1^3h_3$$
$$+240A_2\sigma^2h_0^3h_2h_3-126A_1h_0^2\sigma^2h_1^4h_3+126A_2\sigma^4h_0^4h_2^2h_3+144A_2h_0^2h_1h_3\sigma$$
$$-504A_1h_0^3\sigma^2h_2h_1^2h_3-126A_1h_0^4\sigma^2h_2^2h_3-6A_1h_0^2h_1^2h_3+126A_2\sigma^4h_0^2h_1^4h_3$$
$$-144A_1h_0^3\sigma h_1h_2h_3+720A_2\sigma^3h_0^3h_1h_2h_3-252A_1h_0^4\sigma^2h_3^2h_1)G^5$$
$$+(A_1h_0h_1^4+8A_1h_0^3h_2^2+63A_2\sigma^4h_0^5h_3^2+16A_2\sigma h_0^2h_1h_2+326A_2\sigma^4h_0^4h_1h_2h_3$$
$$+22A_2\sigma^4h_0^3h_1^2h_2^2+18A_2\sigma^2h_0^2h_1^2h_2-A_2h_0h_1^2+324A_2\sigma^2h_0^3h_1h_3+4A_2h_0^2h_2$$
$$+200A_2\sigma^4h_0^3h_1^3h_3+248A_2\sigma^3h_0^4h_2h_3+2A_1h_0^2\sigma^2h_1^4h_2-326A_1h_0^4\sigma^2h_1h_2h_3$$

$$-200A_1h_0^3\sigma^2h_1^3h_3-22A_1h_0^3\sigma^2h_1^2h_2^2+60A_1h_0^3\sigma h_2^2h_1+8A_1h_0^3\sigma h_1^2h_3$$
$$+40A_1h_0^2\sigma h_1^3h_2+72A_2\sigma h_0^3h_3-A_2\sigma^4h_0h_1^6-6A_2\sigma^2h_0h_1^4-4A_2\sigma h_0h_1^3$$
$$+24A_2\sigma^2h_0^3h_2^2+8A_2\sigma^4h_0^4h_2^3+14A_1h_0^4\sigma h_2h_3+4A_2\sigma^3h_0^2h_1^3h_2$$
$$+452A_2\sigma^3h_0^3h_1^2h_3+48A_2\sigma^3h_0^3h_2^2h_1-2A_2\sigma^4h_0^2h_1^4h_2+18A_1h_0^3h_1h_3$$
$$+15A_1h_0^2h_1^2h_2-63A_1h_0^5\sigma^2h_3^2+2A_1h_0\sigma h_1^5+A_1h_0\sigma^2h_1^6-8A_1h_0^4\sigma^2h_2^3$$
$$-4A_2\sigma^3h_0h_1^5)G^4+(8A_1h_0^2h_1^3+12A_1h_0^4h_3-84A_1\sigma^2h_2h_0^5h_3-4A_2\sigma^4h_0^2h_1^5$$
$$+84A_2\sigma^4h_0^5h_2h_3-12A_2\sigma^2h_0^2h_1^3-12A_2\sigma^3h_0^2h_1^4+184A_2\sigma^4h_0^4h_1^2h_3$$
$$+32A_2\sigma^4h_0^4h_1h_2^2+48A_2\sigma^2h_0^3h_1h_2+288A_2\sigma^3h_0^4h_1h_3-32A_1h_0^4\sigma^2h_2^2h_1$$
$$-184A_1h_0^4\sigma^2h_1^2h_3-8A_1h_0^3\sigma^2h_1^3h_2+88A_1h_0^3\sigma h_1^2h_2+48A_1h_0^4\sigma h_1h_3$$
$$+16A_2\sigma h_0^3h_2+108A_2\sigma^2h_0^4h_3+32A_2\sigma^3h_0^4h_2^2+40A_2\sigma^3h_0^3h_1^2h_2+8A_2\sigma^4h_0^3h_1^3h_2$$
$$+28A_1h_0^3h_1h_2+4A_1h_0^2\sigma^2h_1^5+32A_1h_0^4\sigma h_2^2-4A_2\sigma h_0^2h_1^2+12A_1h_0^2\sigma h_1^4)G^3$$
$$+(14A_1h_0^4h_2+18A_1h_0^5\sigma h_3+19A_1h_0^3h_1^2-21A_1h_0^4\sigma^2h_1^2h_2-12A_2\sigma^3h_0^3h_1^3$$
$$-90A_1h_0^5\sigma^2h_1h_3-6A_2\sigma^2h_0^3h_1^2-12A_1h_0^5\sigma^2h_2^2+24A_1h_0^3\sigma h_1^3+66A_1h_0^4\sigma h_1h_2$$
$$+48A_2\sigma^3h_0^4h_1h_2+6A_1h_0^3\sigma^2h_1^4+90A_2\sigma^4h_0^5h_1h_3+72A_2\sigma^3h_0^5h_3+24A_2\sigma^2h_0^4h_2$$
$$+12A_2\sigma^4h_0^5h_2^2-6A_2\sigma^4h_0^3h_1^4+21A_2\sigma^4h_0^4h_1^2h_2)G^2+(-16h_1A_1\sigma^2h_2h_0^5$$
$$+16h_1A_2\sigma^4h_0^5h_2+16A_2\sigma^3h_0^5h_2+18A_1h_1h_0^4+18A_2\sigma^4h_0^6h_3+20A_1\sigma h_1^2h_0^4$$
$$-18A_1h_0^6\sigma^2h_3-4A_2\sigma^4h_0^4h_1^3+4A_1h_0^4\sigma^2h_1^3+16A_1h_0^5\sigma h_2-4A_2\sigma^3h_0^4h_1^2)G$$
$$+A_1h_0^5\sigma^2h_1^2+4A_2\sigma^4h_0^6h_2+6A_1h_0^5\sigma h_1+6A_1h_0^5-A_2\sigma^4h_0^5h_1^2-4A_1h_0^6\sigma^2h_2]. \quad (3.6)$$

Substituting (3.4) and (3.6) into (3.2) we obtain a polynomial in the power of G. Equating the coefficients of this polynomial to zero, we get a system of algebraic equations which can be solved by *Maple* or *Mathematica*. We obtain the results:

$$u_1=A_0+2h_1+\frac{2h_1h_2}{-h_2+C_5h_1\exp\left[-h_1\left(x+y-h_1^2t\right)\right]},$$

$$u_2=A_0+\frac{2C_5h_1^2\left(1+\sigma h_1\right)\exp\left[-h_1\left(x+y-h_1^2t\right)\right]}{-h_2+C_5h_1\left(1+\sigma h_1\right)\exp\left[-h_1\left(x+y-h_1^2t\right)\right]},$$

where $h_1\neq 0$, $h_2\neq 0$ and C_5 are arbitrary constants.

$$u_3=A_0-2h_1-\frac{2h_0h_1}{-h_0+C_4h_1\exp\left[h_1\left(x+y-h_1^2t\right)\right]},$$

$$u_4 = A_0 - \frac{2C_4h_1^2\big(1+\sigma h_1\big)\exp\big[h_1\big(x+y-h_1^2t\big)\big]}{-h_0 + C_4h_1\big(1+\sigma h_1\big)\exp\big[h_1\big(x+y-h_1^2t\big)\big]},$$

where $h_0 \neq 0$, $h_1 \neq 0$ and C_4 are arbitrary constants.

$$u_5 = A_0 + \frac{4C_1h_1^2\exp\big[-2h_1\big(x+y-4h_1^2t\big)\big]}{-h_3 + C_1h_1\exp\big[-2h_1\big(x+y-4h_1^2t\big)\big]},$$

$$u_6 = A_0 + \frac{4C_1h_1^2\big(1+\sigma h_1\big)\exp\big[-2h_1\big(x+y-4h_1^2t\big)\big]}{-h_3 + C_1h_1\big(1+\sigma h_1\big)\exp\big[-2h_1\big(x+y-4h_1^2t\big)\big]},$$

where $h_1 \neq 0$, $h_3 \neq 0$ and C_1 are arbitrary constants.

$$u_7 = A_0 + \frac{4h_3}{C_2 + h_3\big[\sigma - 2\big(x+y-2\sqrt{-3C}t\big)\big]} + \frac{\sqrt{-3C}\,\big\{C_2 + h_3\big[\sigma - 2\big(x+y-2\sqrt{-3C}t\big)\big]\big\}}{6h_3},$$

where $h_3 \neq 0$, and C_2 are arbitrary constants.

$$u_8 = A_0 + \frac{2h_2}{C_6 + h_2\big(\sigma - x - y + 2\sqrt{-3C}t\big)} + \frac{\sqrt{-3C}\,\big[C_6 + h_2\big(\sigma - x - y + 2\sqrt{-3C}t\big)\big]}{3h_2},$$

where $h_2 \neq 0$ and C_6 are arbitrary constants.

$$u_9 = A_0 - \frac{2h_0}{C_7 + h_0\big(\sigma + x + y + 2\sqrt{-3C}t\big)} + \frac{\sqrt{-3C}\,\big[C_7 + h_0\big(\sigma + x + y + 2\sqrt{-3C}t\big)\big]}{3h_0},$$

where $h_0 \neq 0$, and C_7 are arbitrary constants.

3.2. **The $(2+1)$dimensional Burgers equations.** In this section, we consider the following $(2+1)$-dimensional Burgers equations:

$$\begin{gathered} -u_t + uu_y + \alpha v u_x + \beta u_{yy} + \alpha\beta u_{xx} = 0, \\ u_x - v_x = 0, \end{gathered} \tag{3.7}$$

where α and β are nonzero constants. The traveling wave variable (3.7) permits us converting Eq. (3.7) into the following ODE:

$$\begin{gathered} Vu' + uu' + \alpha v u' + \beta(1+\alpha)u'' = 0, \\ u' - v' = 0. \end{gathered} \tag{3.8}$$

Consider the homogeneous balance between u'' with vu' and u'' with uu' in (3.8) we get $n = 1$ and $m = 1$. Using the same idea in Sec. 3.1, we may choose the solution of Equation (3.7) in the form

$$u(\xi) = \sum_{i=-1}^{1} \frac{A_i\big(\frac{G'}{G}\big)^i}{\big[1+\sigma\big(\frac{G'}{G}\big)\big]^i}, \quad v(\xi) = \sum_{i=-1}^{1} \frac{B_i\big(\frac{G'}{G}\big)^i}{\big[1+\sigma\big(\frac{G'}{G}\big)\big]^i}, \tag{3.9}$$

where A_i, B_i and σ are constants to be determined and $G = G(\xi)$ satisfies Eq. (2.5). Substituting (3.4), (3.5) and (3.9) into Eqs. (3.8), we obtain a polynomial in the power of G. On equating the coefficients of this polynomial to zero, we get a system of algebraic equations which can be solved by *Maple* or *Mathematica* and we obtain the following results:

$$u_1 = \beta(1+\alpha)h_1 - V - \alpha B_0 - \frac{2\beta C_5 h_1^2 \exp\left[-h_1(x+y-Vt)\right]}{-h_2 + C_5 h_1 \exp\left[-h_1(x+y-Vt)\right]};$$

$$\upsilon_1 = B_0 - \frac{2\beta C_5 h_1^2 \exp\left[-h_1(x+y-Vt)\right]}{-h_2 + C_5 h_1 \exp\left[-h_1(x+y-Vt)\right]};$$

$$u_2 = A_0 - \frac{2\beta C_5 h_1^2(1+\sigma h_1) \exp\left[-h_1(x+y-Vt)\right]}{-h_2 + C_5 h_1(1+\sigma h_1) \exp\left[-h_1(x+y-Vt)\right]};$$

$$\upsilon_2 = \frac{\beta(1+\alpha)h_1 - V - A_0}{\alpha} - \frac{2\beta C_5 h_1^2(1+\sigma h_1) \exp\left[-h_1(x+y-Vt)\right]}{-h_2 + C_5 h_1(1+\sigma h_1) \exp\left[-h_1(x+y-Vt)\right]},$$

where $h_1 \neq 0$, $h_2 \neq 0$ and C_5 are arbitrary constants.

$$u_3 = 2\beta(1+\alpha)h_1 - V - \alpha B_0 - \frac{4\beta C_1 h_1^2 \exp\left[-2h_1(x+y-Vt)\right]}{-h_3 + C_1 h_1 \exp\left[-2h_1(x+y-Vt)\right]},$$

$$\upsilon_3 = B_0 - \frac{4\beta C_1 h_1^2 \exp\left[-2h_1(x+y-Vt)\right]}{-h_3 + C_1 h_1 \exp\left[-2h_1(x+y-Vt)\right]},$$

$$u_4 = A_0 - \frac{4\beta C_1 h_1^2(1+\sigma h_1) \exp\left[-2h_1(x+y-Vt)\right]}{-h_3 + C_1 h_1(1+\sigma h_1) \exp\left[-2h_1(x+y-Vt)\right]},$$

$$\upsilon_4 = \frac{2\beta(1+\alpha)h_1 - V - A_0}{\alpha} - \frac{4\beta C_1 h_1^2(1+\sigma h_1) \exp\left[-2h_1(x+y-Vt)\right]}{-h_3 + C_1 h_1(1+\sigma h_1) \exp\left[-2h_1(x+y-Vt)\right]},$$

where $h_1 \neq 0$, $h_3 \neq 0$ and C_1 are arbitrary constants.

$$u_5 = -\beta(1+\alpha)h_1 - V - \alpha B_0 + \frac{2\beta C_4 h_1^2 \exp\left[h_1(x+y-Vt)\right]}{-h_0 + C_4 h_1 \exp\left[h_1(x+y-Vt)\right]},$$

$$\upsilon_5 = B_0 + \frac{2\beta C_4 h_1^2 \exp\left[h_1(x+y-Vt)\right]}{-h_0 + C_4 h_1 \exp\left[h_1(x+y-Vt)\right]},$$

$$u_6 = A_0 + \frac{2\beta C_4 h_1^2(1+\sigma h_1) \exp\left[h_1(x+y-Vt)\right]}{-h_0 + C_4 h_1(1+\sigma h_1) \exp\left[h_1(x+y-Vt)\right]},$$

$$\upsilon_6 = -\frac{\beta(1+\alpha)h_1 + V + A_0}{\alpha} + \frac{2\beta C_4 h_1^2(1+\sigma h_1) \exp\left[h_1(x+y-Vt)\right]}{-h_0 + C_4 h_1(1+\sigma h_1) \exp\left[h_1(x+y-Vt)\right]},$$

where $h_0 \neq 0$, $h_1 \neq 0$ and C_4 are arbitrary constants.

Remark 3.1. The ansätz (2.4) and (2.5) proposed in this paper is more general than the ansätz in the original $\left(\frac{G'}{G}\right)$-expansion method [18], the improved $\left(\frac{G'}{G}\right)$-expansion method [5], the generalized $\left(\frac{G'}{G}\right)$-expansion method [10], the extended $\left(\frac{G'}{G}\right)$ -expansion method [6] and the modified $\left(\frac{G'}{G}\right)$-expansion method [2]. If we set the parameters in (2.4) and (2.5) to special values the above methods can be recovered by our proposed method. Therefore our new method is more powerful than the above methods and some new types of traveling wave solutions and solitary wave solutions would be expected for other nonlinear equations.

Remark 3.2. By writing the exact solutions of nonlinear equations as polynomials of $\frac{\left(\frac{G'}{G}\right)}{1+\sigma\left(\frac{G'}{G}\right)}$, the equations can be changed into the nonlinear system of algebraic equations which can be solved with the help of symbolic computation. Therefore our method is a pure algebraic algorithm which can be applied to integrable system and non-integrable system.

Remark 3.3. All solutions presented in this article have been checked with Mathematica by putting them back into the original equations (3.1) and (2.2).

4. CONCLUSIONS

In this work, the enhanced $\left(\frac{G'}{G}\right)$-expansion method has been successfully applied to find the exact solutions of the $(2+1)$-dimensional typical breaking soliton and Burgers equations. In this paper, we investigated the case when $G' = \sum_{i=0}^{3} h_i G^i$. In the future, the proposed method can be extended to the case $i \geq 4$. The present work shows that the enhanced $\left(\frac{G'}{G}\right)$-expansion method is direct, concise and effective, and can be applied to other nonlinear equations in mathematical physics.

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