

ON θ - β -GENERALIZED CLOSED SETS AND θ - β -GENERALIZED CONTINUITY IN TOPOLOGICAL SPACES

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ABSTRACT. Levine [13] introduced and investigated the notion of generalized closed set. It is the object of the present paper to offer a new notion of generalized closed set called θ - β -generalized closed set and study its fundamental properties.

1. INTRODUCTION

In 1970, N. Levine [13] introduced the notion of generalized closed sets. By definition, a subset A of a topological space (X, τ) is called generalized closed set (briefly g -closed) if $\text{Cl}(A) \subset U$ whenever $A \subset U$ and U is open. Noiri [15] and Dontchev and Maki [7] have given independently another new generalization of Levine's g -closed set by utilizing the θ -closure operator. Recall that a subset A of a topological space (X, τ) is θ -generalized closed (briefly θ - g -closed) if $\text{Cl}_\theta(A) \subset U$, whenever $A \subset U$ and U is open in (X, τ) . The concept of θ - g -closed sets was applied to the digital line [8]. Note that, in [15], Noiri used the term generalized θ -closed sets (briefly $g\theta$ -closed sets). We introduce a new form of generalized closed set called θ - β generalized closed set by utilizing β - θ -closure operator. We define the notion of θ -generalized Λ_β -sets and characterize their properties. We also introduce the notion θ - β g -continuity and θ - β g -irresoluteness by using θ - β g -closed sets and study some of their fundamental properties.

2. PRELIMINARIES

Throughout the present paper, spaces (X, τ) and (Y, σ) (or simply, X and Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated. Abd El Monsef et al. [1] and Andrijević [2] introduced the notion of β -open set, which Andrijević called semipreopen, completely independent of each other. In this paper, we adopt the word β -open for the sake of clarity. They characterized the most important properties of β -open sets. A subset A of a topological space (X, τ) is called β -open if $A \subseteq \text{Cl}(\text{Int}(\text{Cl}(A)))$, where $\text{Cl}(A)$ and $\text{Int}(A)$ denote the closure and

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the interior of A , respectively. The complement of a β -open set is called β -closed. The family of all β -open sets of a space X is denoted by $\beta O(X, \tau)$ or $\beta O(X)$. We set $\beta O(X, x) = \{U : x \in U \in \beta O(X)\}$. The intersection of all β -closed sets containing A is called the β -closure of A and is denoted by $\beta Cl(A)$. The β -interior ($\beta Int(A)$) of a subset $A \subset X$ is the union of all β -open sets contained in A .

Lemma 2.1 ([1], [2]). *The following hold for a subset A of a topological space X :*

- (1) $\beta Int(A) = A \cap Cl(Int(Cl(A)))$;
- (2) $\beta Cl(A) = A \cup Int(Cl(Int(A)))$;
- (3) $x \in \beta Cl(A)$ if and only if $O \cap A \neq \emptyset$ for every $O \in \beta O(X, x)$;
- (4) $\beta Cl(X \setminus A) = X \setminus \beta Int(A)$;
- (5) A is β -closed if and only if $A = \beta Cl(A)$.

Now we begin to recall some known notions which will be used in the sequel.

Definition 2.1. [17] Let A be a subset of X . The β - θ -closure of A , $\beta Cl_\theta(A)$, is the set of all $x \in X$ such that $\beta Cl(O) \cap A \neq \emptyset$ for every $O \in \beta O(X, x)$. A is called β - θ -closed if $A = \beta Cl_\theta(A)$. The set $\{x \in X \mid \beta Cl_\theta(O) \subset A \text{ for some } O \in \beta O(X, x)\}$ is called the β θ -interior of A and is denoted by $\beta Int_\theta(A)$. A subset A is called β - θ -open if $A = \beta Int_\theta(A)$.

The following theorem is known and given by Noiri [17].

Theorem 2.1. *For any subset A of X :*

- (1) $\beta Cl_\theta(\beta Cl_\theta(A)) = \beta Cl_\theta(A)$;
- (2) $\beta Cl_\theta(A)$ is β - θ -closed;
- (3) Intersection of arbitrary collection of β - θ -closed set in X is β - θ -closed;
- (4) $\beta Cl_\theta(A)$ is the intersection of all β - θ -closed sets each containing A ;
- (5) If $A \in \beta O(X)$, then $\beta Cl(A) = \beta Cl_\theta(A)$.

For other advances on topological spaces obtained by our research group we recommend [4, 5, 11].

3. θ - β -GENERALIZED CLOSED SETS

Definition 3.1. A subset A of a topological space (X, τ) is called θ - β -generalized closed (briefly θ - β g-closed) if $\beta Cl_\theta(A) \subset O$ whenever $A \subset O$ and $O \in \beta O(X, \tau)$.

The complement of a θ - β -generalized closed set is called θ - β -generalized open (briefly θ - β g-open).

Lemma 3.1. *The following statements hold true (in each case, the converse implications are not true):*

- (i) Every β - θ -closed set is θ - β g-closed.
- (ii) [17] Every β - θ -closed set is β -closed.

Proof. (i) Let $A \subset X$ be β - θ -closed set. Then $A = \beta Cl_\theta(A)$. Let $A \subset O$ and $O \in \beta O(X, \tau)$. It follows that $\beta Cl_\theta(A) \subset O$. This means that A is θ - β g-closed.

- (ii) See Remark 3.2 in [17]. □

Example 3.1. Let $X = \{a, b, c, d\}$ and let $\tau = \{\emptyset, \{c, d\}, X\}$. If $A = \{a, b, d\}$, then $\beta\text{Cl}(A) = X$ and so A is not β -closed. Hence A is not β - θ -closed. Since X is the only β -open set containing A , A is θ - β -g-closed.

Example 3.2. Let $X = \{a, b, c\}$ and let $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. If consider $A = \{c\}$, then A is a β -closed set which is not β - θ -closed.

A subset A of a topological space (X, τ) is called a β -generalized closed set (briefly β g-closed) if $\beta\text{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in \beta O(X, \tau)$. Note that this notion is particular case of generalized $(m1, m2)$ -closed set, introduced by Noiri [16]. A subset B is said to be β -generalized open (briefly β g-open) in (X, τ) if its complement $B^c = X \setminus B$ is β g-closed in (X, τ) .

Lemma 3.2. *The following statements hold true (in each case, the converse implications are not true):*

- (i) Every θ - β g-closed set is β g-closed.
- (ii) Every β -closed set is β g-closed.

Proof. (i) It is true that $\beta\text{Cl}(A) \subset \beta\text{Cl}_\theta(A)$ for every subset A of (X, τ) .

(ii) Let $A \subset X$, be β -closed. Then $A = \beta\text{Cl}(A)$. Let $A \subset O$ and O be β -open. Then $\beta\text{Cl}(A) \subset O$. Hence A is β g-closed. □

Example 3.3. Let $X = \{a, b, c\}$ and let $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$. Set $A = \{c\}$. A is β -closed since it is closed and therefore β g-closed. Set $O = \{a, c\}$. But $\beta\text{Cl}_\theta(A)$ is not a subset of O , where $O \in \beta O(X, \tau)$. Hence A is not θ - β g-closed.

Remark 3.1. Note that β -closedness is independent of θ - β g-closedness as it is showed in the Example 3.1 and Example 3.3.

Lemma 3.3. *Let A be a β -open subset of a topological space (X, τ) . The set A is θ - β g-closed if and only if A is β g-closed.*

Proof. It follows from the fact that $\beta\text{Cl}(A) = \beta\text{Cl}_\theta(A)$ for every $A \in \beta O(X, \tau)$ (Theorem 2.1, [5]). □

Recall that a subset A of a topological space (X, τ) is called a θ -generalized closed set (briefly θ -g-closed set) [7] if $\text{Cl}_\theta(A) \subset U$ whenever $A \subset U$ and U is open in (X, τ) .

It should be noted that θ -g-closedness is independent of θ - β g-closedness as it is shown in the following two examples.

Example 3.4. θ - β g-closedness does not imply θ -g-closedness in general. To show this, let us consider the set $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{b\}, \{c\}, \{b, c\}\}$. We have $\beta O(X, \tau) = \{X, \emptyset, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. The β - θ -closed sets of (X, τ) are $\{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$. The set $\{c\}$ is θ - β g-closed since it is β - θ -closed but it is not θ -g-closed.

Example 3.5. θ -g-closedness does not imply θ - β g-closedness in general. Consider $X = \{a, b, c, d, e\}$ and $\tau = \{X, \emptyset, \{c\}, \{a, b\}, \{a, b, c\}\}$. Let $A = \{a, b, c, e\}$ be a

subset of X . Then A is a θ -g-closed set since X is its only open superset. But A is not θ - β g-closed since $\beta\text{Cl}_\theta(A)$ is not a subset of $O = \{a, b, c, e\}$, where $O \in \beta O(X, \tau)$.

Lemma 3.4. *Let A be a β g-closed subset of (X, τ) . Then $\beta\text{Cl}(A) \setminus A$ does not contain a nonempty β -closed set.*

Proof. Let F be a β -closed set such that $F \subset \beta\text{Cl}(A) \setminus A$. Clearly $A \subset F^c$, where A is β g-closed and F^c is β -open. Thus $\beta\text{Cl}(A) \subset F^c$, or equivalently $F \subset (\beta\text{Cl}(A))^c$. Since by assumption $F \subset \beta\text{Cl}(A)$, then $F \subset (\beta\text{Cl}(A))^c \cap \beta\text{Cl}(A) = \emptyset$. This shows that F coincides with the void-set. \square

Recall (Definition 3.1) that the complement of a θ - β -generalized closed set is called θ - β -generalized open (briefly θ - β g-open).

Theorem 3.1. *A set $A \subset (X, \tau)$ is θ - β g-open if and only if $F \subset \beta\text{Int}_\theta(A)$ whenever F is β -closed in X and $F \subset A$.*

Proof. Necessity. Let A be θ - β g-open and $F \subset A$, where F is β -closed. It is obvious that A^c is contained in F^c . This implies that $\beta\text{Cl}_\theta(A^c) \subset F^c$. Hence $\beta\text{Cl}_\theta(A^c) = (\beta\text{Int}_\theta(A))^c \subset F^c$, i.e. $F \subset \beta\text{Int}_\theta(A)$.

Sufficiency. If F is a β -closed set with $F \subset \beta\text{Int}_\theta(A)$ whenever $F \subset A$, then it follows that $A^c \subset F^c$ and $(\beta\text{Int}_\theta(A))^c \subset F^c$, that is $\beta\text{Cl}_\theta(A^c) \subset F^c$. Therefore A^c is θ - β g-closed and A is θ - β g-open. \square

Recall that a topological space (X, τ) is β - $T_{\frac{1}{2}}$ [3] if and only if every β g-closed set is β -closed.

Next we have some new characterizations of β - $T_{1/2}$ spaces.

Theorem 3.2. *For a topological space (X, τ) the following conditions are equivalent:*

- (i) X is β - $T_{1/2}$.
- (ii) For each $x \in X$, $\{x\}$ is β -closed or β -open.

Proof. (i) \Rightarrow (ii): Suppose that for some $x \in X$, $\{x\}$ is not β -closed. Then $\{x\}^c$ is not β -open. Since X is the only β -open containing $\{x\}^c$, the set $\{x\}^c$ is β g-closed and so it is β -closed in the β - $T_{1/2}$ space (X, τ) . Therefore $\{x\}^c$ is β -closed or equivalently $\{x\}$ is β -open.

(ii) \Rightarrow (i): Let $A \subset X$ be β g-closed. Let $x \in \beta\text{Cl}(A)$. We will show that $x \in A$. By the hypothesis, the singleton $\{x\}$ is either β -closed or β -open. We consider these two cases.

Case 1. $\{x\}$ is β -closed: Then if $x \notin A$, there exists a β -closed set in $\beta\text{Cl}(A) \setminus A$. By Lemma 3.4 $x \in A$.

Case 2. $\{x\}$ is β -open: Since $x \in \beta\text{Cl}(A)$, then $\{x\} \cap A \neq \emptyset$. Thus $x \in A$.

Hence in both cases, we have $x \in A$, that is $\beta\text{Cl}(A) \subset A$ or equivalently A is β -closed since the inclusion $A \subset \beta\text{Cl}(A)$ is trivial. \square

Remark 3.2. $\beta O(X, \tau) \subset \beta GO(X, \tau)$, where $\beta GO(X, \tau)$ is the family of all β g-open sets of (X, τ) .

Let A be β -open. Then A^c is β -closed and β g-closed (Lemma 3.2). This implies A is β g-open. Hence $\beta O(X, \tau) \subset \beta GO(X, \tau)$.

Theorem 3.3. *A topological space (X, τ) is β - $T_{1/2}$ if and only if holds the equality $\beta O(X, \tau) = \beta GO(X, \tau)$.*

Proof. Necessity. Let (X, τ) be β - $T_{1/2}$ topological space. Let $A \in \beta GO(X, \tau)$. Then A^c is β g-closed. By hypothesis, A^c is β -closed and thus $A \in \beta O(X, \tau)$. Hence $\beta O(X, \tau) = \beta GO(X, \tau)$.

Sufficiency. Let $\beta O(X, \tau) = \beta GO(X, \tau)$. Let A be β g-closed. Then A^c is β g-open. Hence $A^c \in \beta O(X, \tau)$. Thus A is β -closed. Therefore (X, τ) is β - $T_{1/2}$. \square

Theorem 3.4. *(X, τ) is a β - $T_{\frac{1}{2}}$ topological space if and only if every θ - β g-closed set is β -closed.*

Proof. Necessity. Let $A \subset X$ be a θ - β g-closed. By Lemma 3.2, A is β g-closed. Since X is a β - $T_{\frac{1}{2}}$ space, A is β -closed.

Sufficiency. Let $x \in X$. If $\{x\}$ is not β -closed, then $\{x\}^c$ is not β -open and thus the only superset of $\{x\}^c$ is X . Trivially $\{x\}^c$ is θ - β g-closed. By hypothesis $\{x\}^c$ is β -closed or equivalently $\{x\}$ is β -open. Hence X is a β - $T_{\frac{1}{2}}$ space. \square

Lemma 3.5. *Let A be a θ - β g-closed subset of (X, τ) . Then,*

- (i) $\beta Cl_{\theta}(A) \setminus A$ does not contain a nonempty β -closed set.
- (ii) $\beta Cl_{\theta}(A) \setminus A$ is θ - β g-open.

Proof. (i) Let F be a β -closed set such that $F \subset \beta Cl_{\theta}(A) \setminus A$. Since F^c is β -open and $A \subset F^c$, it follows that $\beta Cl_{\theta}(A) \subset F^c$, i.e. $F \subset (\beta Cl_{\theta}(A))^c$. This implies that $F \subset (\beta Cl_{\theta}(A))^c \cap \beta Cl_{\theta}(A) = \emptyset$.

(ii) If A is θ - β g-closed and F is a β -closed set such that $F \subset \beta Cl_{\theta}(A) \setminus A$, then by (i), F is empty and therefore $F \subset \beta Int_{\theta}(\beta Cl_{\theta}(A) \setminus A)$. By Theorem 3.1, $\beta Cl_{\theta}(A) \setminus A$ is θ - β g-open. \square

Lemma 3.6. *If the set A is θ - β g-closed set of a topological space (X, τ) such that $A \subset B \subset \beta Cl_{\theta}(A)$, then B is also a θ - β g-closed set of (X, τ) .*

Proof. Let O be a β -open set of the topological space (X, τ) such that $B \subset O$. Then $A \subset O$. Since A is θ - β g-closed, we obtain $\beta Cl_{\theta}(A) \subset O$. By using Theorem 2.1 $\beta Cl_{\theta}(B) \subset \beta Cl_{\theta}(\beta Cl_{\theta}(A)) = \beta Cl_{\theta}(A) \subset O$. Therefore B is also a θ - β g-closed set of (X, τ) . \square

Recall that in a topological space (X, τ) a subset A of a topological space (X, τ) is called a Λ_{β} -set if $A = A^{\Lambda_{\beta}}$, where $A^{\Lambda_{\beta}} = \cap \{O \mid A \subset O, O \in \beta O(X, \tau)\}$ [6].

We now define a generalized Λ_{β} -set and θ -generalized Λ_{β} -set as follows:

Definition 3.2. (i) A subset A of a topological space (X, τ) is called a *generalized Λ_{β} -set* (briefly g. Λ_{β} -set) if $A^{\Lambda_{\beta}} \subset F$ whenever $A \subset F$ and F is a β -closed set of X .
 (ii) A subset A of a topological space (X, τ) is called *θ -generalized Λ_{β} -set* (briefly

θ - g - Λ_β -set) if $A_\theta^{\Lambda_\beta} \subset F$, whenever $A \subset F$ and F is a β -closed set of (X, τ) , where $A_\theta^{\Lambda_\beta} = \{x \in X \mid \beta\text{Cl}_\theta(\{x\}) \cap A \neq \emptyset\}$

Proposition 3.1. *Let $\beta\text{Ker}_\theta(A) = \bigcap \{O \mid A \subset O, O \text{ is } \beta\text{-}\theta\text{-open}\}$ for any subset $A \subset (X, \tau)$. Then $A_\theta^{\Lambda_\beta} = \beta\text{Ker}_\theta(A)$.*

Proof. Let $x \in \beta\text{Ker}_\theta(A)$ and suppose that $x \notin A_\theta^{\Lambda_\beta}$. Then $\beta\text{Cl}_\theta(\{x\}) \cap A = \emptyset$. Hence $x \notin X \setminus \beta\text{Cl}_\theta(\{x\})$, where $X \setminus \beta\text{Cl}_\theta(\{x\})$ is β - θ -open set containing A by Theorem 2.1. But this is impossible since $x \in \beta\text{Ker}_\theta(A)$. Consequently $x \in A_\theta^{\Lambda_\beta}$. Now, if $x \in A_\theta^{\Lambda_\beta}$ and $x \notin \beta\text{Ker}_\theta(A)$, then there exists a β - θ -open set O containing A such that $x \notin O$. Assume that $y \in \beta\text{Cl}_\theta(\{x\}) \cap A$. Thus $y \in O$ and $x \notin O$. But this is a contradiction and we obtain the claim. \square

Proposition 3.2. *The following statements hold true:*

- (i) *For any set $A \subset X$, $A \subset A^{\Lambda_\beta} \subset A_\theta^{\Lambda_\beta} \subset \beta\text{Cl}_\theta(A)$.*
- (ii) *Every β - θ -closed set is a Λ_β -set.*
- (iii) *Every β g -closed Λ_β -set is β -closed.*
- (iv) *Every θ - g - Λ_β -set is a g - Λ_β -set.*

Proof. (i) Let A be a subset of (X, τ) . It is clear that $A \subset A^{\Lambda_\beta}$. Now we prove that $A^{\Lambda_\beta} \subset A_\theta^{\Lambda_\beta}$. Suppose that x is not a point of $A_\theta^{\Lambda_\beta}$. It follows that $A \subset X \setminus \beta\text{Cl}_\theta(\{x\})$ ($= O$, say). Since $\beta\text{Cl}_\theta(\{x\})$ is β - θ -closed by Theorem 2.1, so O is β - θ -open. Hence by Lemma 3.1 O is β -open. Therefore, there exists a β -open set O containing A but not x , i.e. $x \notin A^{\Lambda_\beta}$. This shows that $A^{\Lambda_\beta} \subset A_\theta^{\Lambda_\beta}$.

Proving $A_\theta^{\Lambda_\beta} \subset \beta\text{Cl}_\theta(A)$, let $x \in A_\theta^{\Lambda_\beta}$. Suppose that $x \notin \beta\text{Cl}_\theta(A)$. Then, there exists a β -open set O containing x such that $\beta\text{Cl}(O) \cap A = \emptyset$. Since O is a β -open subset of X , it follows that $\beta\text{Cl}_\theta(O) = \beta\text{Cl}(O)$. Hence $\beta\text{Cl}_\theta(O) \cap A = \emptyset$. This implies that $A \subset X \setminus \beta\text{Cl}_\theta(O)$. Therefore $x \notin X \setminus \beta\text{Cl}_\theta(O)$, where $X \setminus \beta\text{Cl}_\theta(O)$ is a β - θ -open set containing A . But this is impossible since $x \in A_\theta^{\Lambda_\beta}$ by Proposition 3.1. Consequently $x \in \beta\text{Cl}_\theta(A)$.

(ii) Since $A^{\Lambda_\beta} \subset \beta\text{Cl}_\theta(A)$ and $A = \beta\text{Cl}_\theta(A)$, then $A^{\Lambda_\beta} = A$. Thus A is a Λ_β -set.

(iii) Let A be a Λ_β -set β g -closed subset of X . Suppose that $x \notin A_\theta^{\Lambda_\beta}$. So there exists a β -open subset O such that $A \subseteq O$ with $x \notin O$. This means that $x \notin \beta\text{Cl}(A)$ since A is β g -closed. Then $\beta\text{Cl}(A) \subset A^{\Lambda_\beta} = A$. This means that A is β -closed.

(iv) Let $A \subset F$, where F is β -closed. Then we have $A^{\Lambda_\beta} \subset A_\theta^{\Lambda_\beta} \subset F$. This implies that A is a g - Λ_β -set. \square

Lemma 3.7. *Let (X, τ) be a topological space and $x \in X$. The the following two statements are equivalent:*

- (1) $y \in \beta\text{Ker}_\theta(\{x\})$;
- (2) $x \in \beta\text{Cl}_\theta(\{y\})$.

Proof. Let $y \notin \beta\text{Ker}_\theta(\{x\})$. It follows that there exists a β - θ -open set U containing x such that $y \notin U$. This means that $x \notin \beta\text{Cl}_\theta(\{y\})$. The converse can be proved by the same token. \square

Remark 3.3. By Proposition 3.1, we get $\beta\text{Ker}_\theta(A) = \{x \in X \mid \beta\text{Cl}_\theta(\{x\}) \cap A \neq \emptyset\}$.

Lemma 3.8. *The following statements are equivalent for any points x and y in a topological space (X, τ) :*

- (1) $\beta\text{Ker}_\theta(\{x\}) \neq \beta\text{Ker}_\theta(\{y\})$;
- (2) $\beta\text{Cl}_\theta(\{x\}) \neq \beta\text{Cl}_\theta(\{y\})$.

Proof. (1) \Rightarrow (2): Let $\beta\text{Ker}_\theta(\{x\}) \neq \beta\text{Ker}_\theta(\{y\})$. Then there exists a point z in X such that $z \in \beta\text{Ker}_\theta(\{x\})$ and $z \notin \beta\text{Ker}_\theta(\{y\})$. By $z \in \beta\text{Ker}_\theta(\{x\})$, it follows that $\{x\} \cap \beta\text{Cl}_\theta(\{z\}) \neq \emptyset$. This implies $x \in \beta\text{Cl}_\theta(\{z\})$. By $z \notin \beta\text{Ker}_\theta(\{y\})$, we obtain $\{y\} \cap \beta\text{Cl}_\theta(\{z\}) = \emptyset$. Since $x \in \beta\text{Cl}_\theta(\{z\})$, $\beta\text{Cl}_\theta(\{x\}) \subset \beta\text{Cl}_\theta(\{z\})$ and $\{y\} \cap \beta\text{Cl}_\theta(\{x\}) = \emptyset$. Hence it follows that $\beta\text{Cl}_\theta(\{x\}) \neq \beta\text{Cl}_\theta(\{y\})$. Now $\beta\text{Ker}_\theta(\{x\}) \neq \beta\text{Ker}_\theta(\{y\})$ implies that $\beta\text{Cl}_\theta(\{x\}) \neq \beta\text{Cl}_\theta(\{y\})$.

(2) \Rightarrow (1): Let $\beta\text{Cl}_\theta(\{x\}) \neq \beta\text{Cl}_\theta(\{y\})$. Then there exists a point z in X such that $z \in \beta\text{Cl}_\theta(\{x\})$ and $z \notin \beta\text{Cl}_\theta(\{y\})$. This means that there exists a β - θ -open set containing z and therefore x but not y , such that $y \notin \beta\text{Ker}_\theta(\{x\})$. Hence $\beta\text{Ker}_\theta(\{x\}) \neq \beta\text{Ker}_\theta(\{y\})$. \square

Definition 3.3. A topological space (X, τ) is β - θ - R_0 space if every β - θ -open set contains the β - θ -closure of each of its singletons.

The space in Example 3.4 is a β - θ - R_0 space.

Theorem 3.5. *A topological space (X, τ) is a β - θ - R_0 space if and only for any x and y in X , $\beta\text{Cl}_\theta(\{x\}) \neq \beta\text{Cl}_\theta(\{y\}) \Rightarrow \beta\text{Cl}_\theta(\{x\}) \cap \beta\text{Cl}_\theta(\{y\}) = \emptyset$.*

Proof. Suppose that (X, τ) is β - θ - R_0 and $x, y \in X$ such that $\beta\text{Cl}_\theta(\{x\}) \neq \beta\text{Cl}_\theta(\{y\})$. Then, there exists $z \in \beta\text{Cl}_\theta(\{x\})$ such that $z \notin \beta\text{Cl}_\theta(\{y\})$ (or $z \in \beta\text{Cl}_\theta(\{y\})$ such that $z \notin \beta\text{Cl}_\theta(\{x\})$). There exists $V \in \beta\mathcal{O}(X, \tau)$ such that $y \notin V$ and $z \in V$; hence $x \in V$. Therefore, we have $x \notin \beta\text{Cl}_\theta(\{y\})$. Thus $x \in X \setminus \beta\text{Cl}_\theta(\{y\})$, which implies $\beta\text{Cl}_\theta(\{x\}) \subset X \setminus \beta\text{Cl}_\theta(\{y\})$ and $\beta\text{Cl}_\theta(\{x\}) \cap \beta\text{Cl}_\theta(\{y\}) = \emptyset$. The proof for the other case is similar.

Sufficiency. Let V be β - θ -open and let $x \in V$. We will show that $\beta\text{Cl}_\theta(\{x\}) \subset V$. Let $y \notin V$, such that $y \in X \setminus V$. Then $x \neq y$ and $x \notin \beta\text{Cl}_\theta(\{y\})$. This shows that $\beta\text{Cl}_\theta(\{x\}) \neq \beta\text{Cl}_\theta(\{y\})$. By assumption $\beta\text{Cl}_\theta(\{x\}) \cap \beta\text{Cl}_\theta(\{y\}) = \emptyset$. Hence $y \notin \beta\text{Cl}_\theta(\{x\})$. Therefore $\beta\text{Cl}_\theta(\{x\}) \subset V$. \square

Theorem 3.6. *A topological space (X, τ) is a β - θ - R_0 space if and only if for any points x and y in X , $\beta\text{Ker}_\theta(\{x\}) \neq \beta\text{Ker}_\theta(\{y\}) \Rightarrow \beta\text{Ker}_\theta(\{x\}) \cap \beta\text{Ker}_\theta(\{y\}) = \emptyset$.*

Proof. Suppose that (X, τ) is an β - θ - R_0 space. By Lemma 3.8, we find out that for any points x and y in X , if $\beta\text{Ker}_\theta(\{x\}) \neq \beta\text{Ker}_\theta(\{y\})$, then $\beta\text{Cl}_\theta(\{x\}) \neq \beta\text{Cl}_\theta(\{y\})$. Now we prove that $\beta\text{Ker}_\theta(\{x\}) \cap \beta\text{Ker}_\theta(\{y\}) = \emptyset$. In this respect, we assume that $z \in \beta\text{Ker}_\theta(\{x\}) \cap \beta\text{Ker}_\theta(\{y\})$. By $z \in \beta\text{Ker}_\theta(\{x\})$ and Lemma 3.7, it follows that

$x \in \beta\text{Cl}_\theta(\{z\})$. Since $x \in \beta\text{Cl}_\theta(\{x\})$, by Theorem 3.5 $\beta\text{Cl}_\theta(\{x\}) = \beta\text{Cl}_\theta(\{z\})$. Similarly, we have $\beta\text{Cl}_\theta(\{y\}) = \beta\text{Cl}_\theta(\{z\}) = \beta\text{Cl}_\theta(\{x\})$. This is a contradiction. Therefore, we have $\beta\text{Ker}_\theta(\{x\}) \cap \beta\text{Ker}_\theta(\{y\}) = \emptyset$.

To prove the converse, let (X, τ) be a topological space such that for any points x and y in X , $\beta\text{Ker}_\theta(\{x\}) \neq \beta\text{Ker}_\theta(\{y\})$ implies $\beta\text{Ker}_\theta(\{x\}) \cap \beta\text{Ker}_\theta(\{y\}) = \emptyset$. If $\beta\text{Cl}_\theta(\{x\}) \neq \beta\text{Cl}_\theta(\{y\})$, then by Lemma 3.8, $\beta\text{Ker}_\theta(\{x\}) \neq \beta\text{Ker}_\theta(\{y\})$. Hence $\beta\text{Ker}_\theta(\{x\}) \cap \beta\text{Ker}_\theta(\{y\}) = \emptyset$ which implies $\beta\text{Cl}_\theta(\{x\}) \cap \beta\text{Cl}_\theta(\{y\}) = \emptyset$. Now, because $z \in \beta\text{Cl}_\theta(\{x\})$ implies $x \in \beta\text{Ker}_\theta(\{z\})$, therefore $\beta\text{Ker}_\theta(\{x\}) \cap \beta\text{Ker}_\theta(\{z\}) \neq \emptyset$. By hypothesis, we have $\beta\text{Ker}_\theta(\{x\}) = \beta\text{Ker}_\theta(\{z\})$. Then $z \in \beta\text{Cl}_\theta(\{x\}) \cap \beta\text{Cl}_\theta(\{y\})$ implies $\beta\text{Ker}_\theta(\{x\}) = \beta\text{Ker}_\theta(\{z\}) = \beta\text{Ker}_\theta(\{y\})$ and this is a contradiction. Hence, $\beta\text{Cl}_\theta(\{x\}) \cap \beta\text{Cl}_\theta(\{y\}) = \emptyset$ and by Theorem 3.25 (X, τ) is a β - θ - R_0 space. \square

4. θ - β GENERALIZED CONTINUOUS FUNCTIONS

Definition 4.1. A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called

(i) θ - β *generalized continuous* (briefly θ - β g-continuous) if $f^{-1}(F)$ is θ - β g-closed in (X, τ) for every β -closed set F of (Y, σ) .

(ii) θ - β *generalized irresolute* (briefly θ - β g-irresolute) if $f^{-1}(F)$ is θ - β g-closed in (X, τ) for every θ - β g-closed set F of (Y, σ) .

Note that a function $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be *pre β -open* [14] (resp. *pre β -closed* [14]) if $f(U) \in \beta O(Y, \sigma)$ for every $U \in \beta O(X, \tau)$ (resp. if $f(U)$ is β -closed for every β -closed set U).

Theorem 4.1. *If $f: (X, \tau) \rightarrow (Y, \sigma)$ is a bijective pre β -open and θ - β g-continuous function, then f is θ - β g-irresolute.*

Proof. Let F be a θ - β g-closed set of (Y, σ) . Assume that $f^{-1}(F) \subset O$, where $O \in \beta O(X, \tau)$. Clearly $F \subset f(O)$. Since $f(O) \in \beta O(Y, \sigma)$ and F is a θ - β g-closed set, we have $\beta\text{Cl}_\theta(F) \subset f(O)$ and therefore $f^{-1}(\beta\text{Cl}_\theta(F)) \subset O$. By the fact that f is θ - β g-continuous and $\beta\text{Cl}_\theta(F)$ is β -closed (Lemma 3.1), it follows that $\beta\text{Cl}_\theta(f^{-1}(\beta\text{Cl}_\theta(F))) \subset O$ and thus $\beta\text{Cl}_\theta(f^{-1}(F)) \subset O$. This means that $f^{-1}(F)$ is θ - β g-closed in (X, τ) and therefore f is θ - β g-irresolute. \square

Corollary 4.1. *If $f: (X, \tau) \rightarrow (Y, \sigma)$ is a bijective pre β -open θ - β g-continuous function and (X, τ) is β - $T_{\frac{1}{2}}$, then (Y, σ) is β - $T_{\frac{1}{2}}$.*

Proof. It suffices to apply Theorem 3.4 and Theorem 4.1. \square

Definition 4.2. A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be θ - β g-closed if for every β -closed set F of (X, τ) , $f(F)$ is θ - β g-closed in (Y, σ) .

Recall that a function $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be β -irresolute [14] if $f^{-1}(V) \in \beta O(X, \tau)$ for every $V \in \beta O(Y, \sigma)$.

Theorem 4.2. *If $f: (X, \tau) \rightarrow (Y, \sigma)$ is β -irresolute and θ - β g-closed, then $f(A)$ is θ - β g-closed in Y for a θ - β g-closed set A of X .*

Proof. Let A be a θ - β g-closed set of X . Assume that $f(A) \subset O$ where $O \in \beta O(Y)$. Clearly $A \subset f^{-1}(O)$. Since $f^{-1}(O) \in \beta O(X)$ and A be a θ - β g-closed set, we have $\beta Cl_{\theta}(A) \subset f^{-1}(O)$ and therefore $f(\beta Cl_{\theta}(A)) \subset O$. Since f is θ - β g-closed and $\beta Cl_{\theta}(A)$ is β -closed, it follows that $\beta Cl_{\theta}(f(\beta Cl_{\theta}(A))) \subset O$ and thus $\beta Cl_{\theta}(f(A)) \subset O$. Thus means that $f(A)$ is θ - β g-closed. \square

Definition 4.3. A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be *quasi- β -irresolute* [17] (resp. *strongly β -irresolute* [17]) if for each $x \in X$ and each $V \in \beta O(Y, f(x))$, there exists $U \in \beta O(X, x)$ such that $f(U) \subset \beta Cl(V)$ (resp. $f(\beta Cl(U)) \subset V$).

Note that, in [17], quasi- β -irresolute function is defined as weakly- β -irresolute.

Theorem 4.3. [17] *Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a function, then:*

- i) *f is quasi- β -irresolute if and only if $f^{-1}(V)$ is β - θ -closed (resp. β - θ -open) in (X, τ) for every β - θ -closed (resp. β - θ -open) set V in (Y, σ)*
- ii) *f is quasi- β -irresolute if and only if $f^{-1}(V)$ is β -closed in (X, τ) for every β - θ -closed set V in (Y, σ)*
- iii) *f is strongly- β -irresolute if and only if $f^{-1}(V)$ is β - θ -closed (resp. β - θ -open) in (X, τ) for every β -closed (resp. β -open) set V in (Y, σ)*

Theorem 4.4. *If the function $f: (X, \tau) \rightarrow (Y, \sigma)$ is strongly β -irresolute then f is θ - β g-continuous.*

Theorem 4.5. *Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be θ - β g-irresolute function. Then f is quasi- β -irresolute if (X, τ) is β - $T_{\frac{1}{2}}$.*

Proof. Suppose that V is a β - θ -closed set in (Y, σ) . By Lemma 3.1, V is θ - β g-closed in (Y, σ) . Since f is θ - β g-irresolute, $f^{-1}(V)$ is θ - β g-closed in (X, τ) . By Theorem 3.4, $f^{-1}(V)$ is β -closed. This shows that f is quasi- β -irresolute (Theorem 4.3). \square

Definition 4.4. A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called *pre- β - θ -open* (resp. *pre- β - θ -closed*) if $f(U)$ is a β - θ -open (resp. β - θ -closed) set of (Y, σ) for every β - θ -open (resp. β - θ -closed) set U of (X, τ) .

Theorem 4.6. *If a function $f: (X, \tau) \rightarrow (Y, \sigma)$ is quasi- β -irresolute and pre- β -closed, then for every θ - β g-closed set F of (Y, σ) , $f^{-1}(F)$ is θ - β g-closed set of (X, τ) (i.e., f is θ - β g-irresolute).*

Proof. Suppose the set F is a θ - β g-closed set of (Y, σ) . Assume $f^{-1}(F) \subset O$, where $O \in \beta O(X, \tau)$. By hypothesis, the function f is quasi- β -irresolute and therefore $f(\beta Cl_{\theta}(f^{-1}(F)) \cap O^c) \subset \beta Cl_{\theta}(f(f^{-1}(F))) \cap f(f^{-1}(F^c)) \subset \beta Cl_{\theta}(F) \setminus F$. By the fact that f is pre- β -closed, it follows that $\beta Cl_{\theta}(F) \setminus F$ contains a β -closed subset $f(\beta Cl_{\theta}(f^{-1}(F)) \cap O^c)$. Now we have $f(\beta Cl_{\theta}(f^{-1}(F)) \cap O^c) = \emptyset$ by Lemma 3.5. This implies that $\beta Cl_{\theta}(f^{-1}(F)) \subset O$. Therefore, this shows that $f^{-1}(F)$ is θ - β g-closed set of (X, τ) . \square

Theorem 4.7. *If a function $f: (X, \tau) \rightarrow (Y, \sigma)$ is β -irresolute and pre- β - θ -closed, then for every θ - β g-closed set D of (X, τ) , $f(D)$ is θ - β g-closed set of (Y, σ) .*

Proof. Suppose that D is a θ - β g-closed set of (X, τ) . Assume $f(D) \subset O$, where $O \in \beta O(Y, \sigma)$. Now $D \subset f^{-1}(O)$ and since f is β -irresolute $f^{-1}(O) \in \beta O(X, \tau)$. But D is θ - β g-closed and therefore $\beta \text{Cl}_\theta(D) \subset f^{-1}(O)$. Thus $f(\beta \text{Cl}_\theta(D)) \subset O$. Now we have $\beta \text{Cl}_\theta(f(D)) \subset \beta \text{Cl}_\theta(f(\beta \text{Cl}_\theta(D))) = f(\beta \text{Cl}_\theta(D)) \subset O$. This shows that $f(D)$ is a θ - β g-closed set of (Y, σ) . \square

Theorem 4.8. *If a topological space (X, τ) is β - $T_{\frac{1}{2}}$ and $f: (X, \tau) \rightarrow (Y, \sigma)$ is surjective, quasi- β -irresolute and pre- β -closed, then (Y, σ) is β - $T_{\frac{1}{2}}$.*

Proof. Assume that A is a θ - β g-closed subset of (Y, σ) . Then by Theorem 4.6 we have $f^{-1}(A)$ is a θ - β g-closed subset of (X, τ) . By Theorem 3.4, $f^{-1}(A)$ is β -closed and hence, A is β -closed. It follows that (Y, σ) is β - $T_{\frac{1}{2}}$. \square

Underline that:

i) If $f: (X, \tau) \rightarrow (Y, \sigma)$ is θ - β g-irresolute function and $g: (Y, \sigma) \rightarrow (Z, \mu)$ is θ - β g-continuous, then $g \circ f$ is θ - β g-continuous.

ii) If $f: (X, \tau) \rightarrow (Y, \sigma)$ and $g: (Y, \sigma) \rightarrow (Z, \mu)$ are θ - β g-irresolute, then $g \circ f$ is θ - β g-irresolute.

Recall that a topological space X is said to be β -connected [18] if X can not be written as a disjoint union of two non-empty β -open sets. For other properties of β -connected spaces see [12].

We now have the following,

Definition 4.5. A topological space X is said to be *$T\beta GO$ -connected* (resp. *βGO -connected*) if X can not be written as a disjoint union of two non-empty θ - β g-open (resp. β g-open) sets.

Theorem 4.9. *For a topological space (X, τ) the following conditions are equivalent:*

- (1) X is $T\beta GO$ -connected;
- (2) the only subsets of X both θ - β g-closed and θ - β g-open are \emptyset and X .

Proof. (1) \Rightarrow (2): Suppose that A is a θ - β g-closed and θ - β g-open subset of X . Then the complement of A , i.e. $X \setminus A$ is θ - β g-closed and θ - β g-open. By the fact that $X = A \cup (X \setminus A)$ and $A \cap (X \setminus A) = \emptyset$, either A is empty or $A = X$.

(2) \Rightarrow (1): Let $X = A \cup B$, where A and B are θ - β g-open sets and $A \cap B = \emptyset$. Then A is θ - β g-open and θ - β g-closed. By hypothesis either A is empty or $A = X$. This implies that X is $T\beta GO$ -connected. \square

Theorem 4.10. *If (X, τ) is β - $T_{\frac{1}{2}}$, then the following conditions are equivalent:*

- (1) X is βGO -connected;
- (2) X is $T\beta GO$ -connected;
- (3) X is β -connected.

Example 4.1. Let us remark that the space $X = \{a, b, c\}$ with the topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ is connected. Since $\{b\}$ is both θ - β g-open and θ - β g-closed, (X, τ) is neither $T\beta GO$ -connected nor βGO -connected nor β -connected.

Theorem 4.11. *The following statements hold true:*

(1) *If $f: (X, \tau) \rightarrow (Y, \sigma)$ is a surjective θ - β g-continuous function and X is $T\beta$ GO-connected, then Y is β -connected;*

(2) *If $f: (X, \tau) \rightarrow (Y, \sigma)$ is a surjective θ - β g-irresolute function and the set X is $T\beta$ GO-connected, then Y is $T\beta$ GO-connected.*

Proof. (1) Suppose that Y is not β -connected. There exist non-empty β -open sets V and W of Y such that $V \cup W = Y$ and $V \cap W = \emptyset$. Since f is θ - β g-continuous and surjective $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty and θ - β g-open in X . This contradicts the fact that X is $T\beta$ GO-connected.

(2) The argument is a minor modification of the proof (i). □

5. CONCLUSION

Maps have always been of tremendous importance in all branches of mathematics and the whole science. On the other hand, topology plays a significant role in quantum physics, high energy physics and superstring theory [9, 10]. Thus we have obtained a new class of mappings called θ - β g-continuous which may have possible application in quantum physics, high energy physics and superstring theory.

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