

SOME DENSE SUBSETS OF REAL NUMBERS AND THEIR APPLICATIONS

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ABSTRACT. We give a collection of subsets which are dense in the set of real numbers. Several applications of the dense sets are also presented.

1. INTRODUCTION

Throughout this paper, we use \mathbb{R} , \mathbb{R}^+ , \mathbb{R}^- , \mathbb{Q} , \mathbb{Z} , and \mathbb{Z}^+ to denote real numbers, positive real numbers, negative real numbers, rational numbers, integers, and positive integers, respectively. In [8], He, Sinkala, and Zha present a collection of “small” subsets which are dense in \mathbb{R} . A special case of the result was obtained by Moser and Macon [11] and recently repeated by [2]. They derived a similar result for the integer setting. Moreover, Heuer also gave an alternative proof of Moser and Macon’s result in [9]. However, in the next section, we shall present a much simpler proof of the result, by using the Kroncker’s theorem, instead of the previous elementary method in [8]. One may see this generalizes the result of [11] and [2] to pairs of real numbers and give a necessary and sufficient condition. In the last section, we shall give some applications of the result on the dense sets shown in Section 2. Here and throughout the paper, unless stated otherwise, our logarithmic function $\log x$ is a natural logarithm function (following the notation of [7]).

2. DENSE SUBSETS OF REAL NUMBERS

In this section, we construct a collection of infinitely many dense subsets (see p. 32 of [13] for the definition of dense subsets) of \mathbb{R} that are very “small”. These dense subsets may not see dense in \mathbb{R} at first glance. For example, we show that both $S_{2,3} = \{\pm 2^n 3^m : n, m \in \mathbb{Z}\}$ and $S_{\pi,e} = \{\pm \pi^n e^m : n, m \in \mathbb{Z}\}$ are dense in \mathbb{R} . In general, we prove the following main theorem.

Theorem 2.1. *Suppose p and q are in $\mathbb{R}^+ \setminus \{1\}$. Let*

$$S_{p,q} = \{\pm p^n q^m : n, m \in \mathbb{Z}\}.$$

Received: July 3, 2011. *Revised:* July 16, 2011.

2010 *Mathematics Subject Classification:* 05A16, 11K06, 11B05, 65Q20, 65B10.

Key words and phrases: Dense sets, k -dimensional Kroncker’s theorem, uniform distribution, Benford law, functional equation, Cauchy equation.

Then $S_{p,q}$ is dense in \mathbb{R} if and only if $\log_q p \notin \mathbb{Q}$.

Proof. It is easy to see that $S_{p,q}$ is not dense in \mathbb{R}^+ , if $\log p / \log q$ is rational. Hence, we only need to prove sufficiency.

Let p and $q \in \mathbb{R}^+ \setminus \{1\}$ such that $\log_q p$ be an irrational number. We now prove the case of $q > 1$. The circumstance of $0 < q < 1$ can be proved similarly. Denote $\theta = \log_q p$, and for an arbitrary $x \in \mathbb{R}^+$ let $\alpha = \frac{\log x}{\log q}$. Then by Kronecker's theorem (cf. for example, p. 375, [7] or, p. 82, [12]), for any n and $\epsilon > 0$, there exist integers $a > n$ and $b > 0$ such that

$$|a\theta - \alpha - b| < \epsilon,$$

or equivalently,

$$|a \log p - \log x - b \log q| < \epsilon \log q,$$

which implies that $a \log p - b \log q \in (\log x - \epsilon \log q, \log x + \epsilon \log q)$. Thus $\frac{p^a}{q^b} \in (xq^{-\epsilon}, xq^\epsilon)$. Hence, sufficiency is proved. \square

Remark 2.1. If p is a rational number, $p = r/s$, with $\gcd(r, s) = 1$, then $\log_q p \in \mathbb{Q}$ if and only if $q|r$. Indeed, if $\log_q p = \log_q \frac{r}{s} = \frac{m}{n}$, $\gcd(m, n) = 1$, then $q^{m/n} = r/s$, or equivalently, $s^n q^m = r^n$, which implies $q|r$.

Theorem 2.1 can be extended to a higher dimensional case.

Theorem 2.2. *If $p_1, p_2, \dots, p_k, q \in \mathbb{R}^+ \setminus \{1\}$ such that $\log_q p_i$, $i = 1, 2, \dots, k$, and 1 are linearly independent over \mathbb{Q} , then*

$$S_{p_1, p_2, \dots, p_k, q} := \{(\pm p_1^n q^{m_1}, \pm p_2^n q^{m_2}, \dots, \pm p_k^n q^{m_k}) : n, m_1, m_2, \dots, m_k \in \mathbb{Z}\}$$

is dense in \mathbb{R}^k .

Proof. Theorem 2.2 can be proved using the k -dimensional Kronecker's theorem (cf. [7], p. 382), which is similar to the proof of Theorem 2.1. More precisely, denote $\theta_i = \log_q p_i$ ($i = 1, 2, \dots, k$), and for any $x \in (\mathbb{R}^+)^k$, $x = (x_1, x_2, \dots, x_k)$, denote $\alpha_i = \log_q x_i$ ($i = 1, 2, \dots, k$). Then k -dimensional Kronecker's theorem says for any n and $\epsilon > 0$, there exist $a > n$ and $b_1, b_2, \dots, b_k \in \mathbb{N}$ such that

$$|a\theta_i - \alpha_i - b_i| < \epsilon,$$

or equivalently,

$$|a \log p_i - \log x_i - b_i \log q| < \epsilon \log q.$$

The last inequality implies that $p_i^a / q^{b_i} \in (x_i q^{-\epsilon}, x_i q^\epsilon)$, and this completes the proof. \square

Corollary 2.1. *Suppose $p \in \mathbb{N}$ and $p > 1$. Then $S_{p, p+r}$ is dense in \mathbb{R} for any positive odd integer r .*

Proof. From Theorem 2.1, we only need to show that $\log_p(p+r)$ is not rational for any $p \in \mathbb{N} \setminus \{1\}$ and any positive odd integer r . Actually, if $\log_p(p+r) = a/b$ for some $a, b \in \mathbb{Z}^+$ with $\gcd(a, b) = 1$, then $(p+r)^b = p^a$. This is impossible because the left hand side and right hand side of the equation have different parity. This contradiction shows the correctness of our claim. \square

In addition, we have the following straightforward and useful corollary.

Corollary 2.2. *Suppose $p \in \mathbb{N} \setminus \{1\}$ and r is a positive odd integer. If the linear functional equation denoted by $\mathcal{L}f(x) = 0$ with initial condition $f(x_0) = c$ at point $x_0 \in \mathbb{R}$ has a solution $f(x)$ for all $x \in S_{p,p+r}$, then $f(x)$ is a unique continuous solution of the functional equation for every x in \mathbb{R} .*

Here we give some applications of Corollary 2.2 for solving Cauchy type functional equations. All solutions of the functional equations discussed in the paper are non-trivial, i.e., the solutions are not identically to be zero.

Example 2.1. Besides a constant multiplier c , the linear function $f(x) = cx$ is the unique continuous function that satisfies $f(0) = 0$ and

$$f(x+y) = f(x) + f(y), \quad (2.1)$$

for all $x, y \in \mathbb{R}$. First of all, (2.1) implies $f(0) = 0$ and has solution $f(x) = cx$ for a constant $c \in \mathbb{R}$. It is obvious that we only need to show the conclusion is true for all $x, y \in \mathbb{R}^+$. Thus, Theorem 2.1 tells us that it is sufficient to prove the result in $S_{p,p+1}$ for any positive integer $p > 1$, say 2. If $x = y$, then (2.1) gives $f(2x) = 2f(x)$. Using induction we have $f(2^n x) = 2^n f(x)$. Similarly, we have $f(2^n 3^m x) = 2^n 3^m f(x)$ for all $n, m \in \mathbb{N}$. In $f(2^n x) = 2^n f(x)$, we use transform $x \mapsto 2^{-n}x$ and obtain $f(2^{-n}x) = 2^{-n}f(x)$. We can establish similarly $f(2^n 3^m x) = 2^n 3^m f(x)$ for all $n, m \in \mathbb{Z}$. Thus, Corollary 2.2 implies our conclusion.

Similarly, we have the following examples, in which we do not consider the trivial case of $f(x) \equiv 0$ in the solution of the functional equation.

Example 2.2. The Cauchy's exponential function $f(x) = a^x$ ($a > 0$) is the unique continuous function that satisfies $f(0) = 1$ and

$$f(x+y) = f(x)f(y),$$

for all $x, y \in \mathbb{R}$.

Note that $f(2^n 3^m x) = (f(x))^{2^n 3^m}$, for all $n, m \in \mathbb{Z}$.

Example 2.3. The logarithm function $f(x) = \log_a x$ ($a > 0, \neq 1$) is the unique continuous function that satisfies $f(1) = 0$ and

$$f(xy) = f(x) + f(y),$$

for all $x, y \in \mathbb{R}^+$.

Note that $f(x^{2^n 3^m}) = 2^n 3^m f(x)$, for all $n, m \in \mathbb{Z}$.

Example 2.4. The power function $f(x) = x^\mu$ ($\mu > 0, \neq 1$) is the unique continuous function that satisfies $f(1) = 1$ and

$$f(xy) = f(x)f(y),$$

for all $x, y \in \mathbb{R}^+$.

Note that $f(x^{2^n 3^m}) = (f(x))^{2^n 3^m}$.

In [2], a characteristic property of the polynomial function $y = kx^2$ in \mathbb{R}^+ is presented by using Candido's equation

$$f(f(x) + f(y) + f(x + y)) = 2[f(f(x)) + f(f(y)) + f(f(x + y))]. \quad (2.2)$$

More precisely, the authors proved that a continuous surjective function f from \mathbb{R}^+ to \mathbb{R}^+ such that $f(0) = 0$ satisfies Candido's equation (2.2) if and only if $f(x) = kx^2$, where $k > 0$ is an arbitrary constant. Their proof is based on the fact that

$$f(2^n 3^m) = (2^n 3^m)^2 f(1),$$

for all integers m, n . Then, from the density of $\{2^n 3^m\}_{n, m \in \mathbb{Z}}$ in \mathbb{R}^+ , the conclusion is obtained. It is obvious that this result can be implied by Corollary 2.2.

3. APPLICATIONS AND RELATED TOPICS

Besides two applications shown as in [2] and [8], we shall present more applications and related materials as follows.

Example 3.1. We now discuss an application to the p -Benford. From [5], for $p \in \mathbb{N}$, a sequence $\{x_n\}$ is p -Benford if and only if $\{\log_p |x_n|\}$ is u.d.mod 1. Here, a double sequence s_{jk} , $j, k = 1, 2, \dots$, of real numbers is said to be uniformly distributed modulo 1 (abbreviated u.d.mod 1) if for any a and b such that $0 \leq a < b \leq 1$,

$$\lim_{M, N \rightarrow \infty} \frac{A([a, b]; M, N)}{MN} = b - a,$$

where $A([a, b]; M, N)$ is the number of s_{jk} , $1 \leq j \leq M$, $1 \leq k \leq N$, for which the fraction parts of s_{jk} satisfy $a \leq \{s_{jk}\} < b$. Kuipers and Niederreiter in [10] showed that the double sequence $\{s_{jk}\}$ is u.d.mod 1 if and only if

$$\lim_{M, N \rightarrow \infty} \frac{1}{MN} \sum_{j=1}^M \sum_{k=1}^N e^{2\pi i h s_{jk}} = 0, \quad (3.1)$$

for all integers $h \neq 0$. Let θ be irrational and α an arbitrary real number. Then $(j\theta + k\alpha)$, $j, k = 1, 2, \dots$, is u.d.mod 1, follows easily from criterion (3.1).

Considering $S_{p,q}^+ = \{p^n q^m : n, m \in \mathbb{Z}\}$, a u.d.mod 1 sequence

$$LS_{p,q}^+ := \{\log_{10} p^n q^m : n, m \in \mathbb{N}\} = \{n \log_{10} p + m \log_{10} q : n, m \in \mathbb{N}\}$$

is constructed if $\log_p q$ is irrational. Thus, if p and q are in $\mathbb{R}^+ \setminus \{1\}$ and $\log_q p \notin \mathbb{Q}$, then $LS_{p,q}^+$ is a u.d.mod 1 sequence.

Similarly, double sequence

$$\tilde{L}S_{p,q}^+ := \{\log_p p^n q^m : n, m \in \mathbb{N}\} = \{n + m \log_p q : n, m \in \mathbb{N}\} = \{m \log_p q\}$$

is u.d.mod 1 if $\log_p q$ is irrational. Let $p \in \mathbb{N}$. We have mentioned that Diaconis in [5] shows that a sequence $\{x_n\}$ is p -Benford if and only if $\{\log_p |x_n|\}$ is u.d.mod 1. Here, a sequence of positive numbers $\{x_n\}$ is p -Benford, or equivalently, Benford base p , if the probability of observing the first digit of x_n in base p is j is equal to $\log_p(1 + 1/j)$. More precisely,

$$\lim_{N \rightarrow \infty} \frac{\#\{n \leq N : \text{first digit of } x_n \text{ in base } p \text{ is } j\}}{N} = \log_p \left(1 + \frac{1}{j}\right),$$

where $j \in \{1, \dots, p-1\}$. This is a probability distribution as one of the $p-1$ events must occur, and the total probability is

$$\sum_{j=1}^{p-1} \log_p \left(1 + \frac{1}{j}\right) = \log_p \prod_{j=1}^{p-1} \left(1 + \frac{1}{j}\right) = \log_p p = 1.$$

In $LS_{p,q}^+ := \{\log p^n q^m : n, m \in \mathbb{N}\} = \{n \log p + m \log q : n, m \in \mathbb{N}\}$ denote $x_n = n \log p$. Recall that if $\{x_n\}$ is u.d.mod 1, then for every $\alpha \in \mathbb{R}$ and $b \in \mathbb{N} \setminus \{1\}$, the sequence $\{x_n + \alpha \log_b n\}$ is also u.d.mod 1. Therefore,

$$\begin{aligned} LS_{p,q}^+ &:= \{\log p^n q^m : n, m \in \mathbb{N}\} = \{n \log p + m \log q : n, m \in \mathbb{N}\} \\ &= \left\{ x_n + \left(\frac{m \log p \log q}{\log_p n} \right) \log_p n \right\} \end{aligned}$$

is u.d.mod 1. Therefore, we obtain the following result.

Proposition 3.1. *If $p \in \mathbb{N}$ such that $\log_p q$ is irrational, then*

$$S_{p,q}^+ = \{p^n q^m : n, m \in \mathbb{Z}\}$$

is p -Benford.

It is known that Benford's law makes an important rule in dynamic systems. For instance, one-dimensional dynamical systems with Benford sequences as orbits were studied in [4] and similar questions for higher dimensions by looking at the component sequences were treated in [3].

Example 3.2. In [1], Aczel and Alsina defined the following n -spiral function $F(a, \theta)$ and discuss the conditions under which an n -spiral function is the Archimedean spiral $a\theta$. Given $n \geq 2$, an n -spiral function is a function $F : (\mathbb{R}^+)^2 \mapsto \mathbb{R}^+$ satisfying conditions:

- (i) $F(x + y, \theta) = F(x, \theta) + F(y, \theta)$;
- (ii) $F\left(a, \frac{\theta}{n}\right) = \frac{1}{n}F(a, \theta)$.

For fixed θ , from Corollary 2.2 and Example 2.1, we immediately have $F(x, \theta) = F(1, \theta)x$. Following [1], if we denote $F(1, \theta) = \phi(\theta)$, then $\phi(\theta)$ satisfies $\phi(x/n) = \phi(x)/n$ from (ii). In addition, ϕ satisfies $\phi(0) = 0$. Conversely, if

$F(x, \theta) = x\phi(\theta)$, where $\phi(0) = 0$ and $\phi(\theta)$ satisfies $\phi(x/n) = \phi(x)/n$, then we have (i) and (ii). This is exactly Theorem 1 of [1]. Furthermore, if F is a continuous n -spiral function for both $n = p$ and $n = q$, where $p, q \in \mathbb{N}$ such that $\log_p q$ is irrational, then from Corollary 2.2, we have $F(a, \theta) = a\theta$, the Archimedean spiral.

Example 3.3. In [6], Einsiedler described a highly interesting question raised by Furstenberg around 1967. Furstenberg showed the orbit set $\{2^k 3^\ell(x) : k, \ell \geq 1\}$ is dense in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ whenever the starting point $x \in \mathbb{T}/\mathbb{Q}$ is irrational. Here, the orbit is taken with respect to the semigroup generated by maps $T_2(x) = 2x$ and $T_3(x) = 3x$ for $x \in \mathbb{T}$. Since there is often a correspondence between orbits and invariant measures (see [6]), one may ask the following question: What are the probability measures on \mathbb{T} that are at the same time invariant under T_2 and T_3 ?

Example 3.4. The question about the distribution of $(3/2)^n \bmod 1$ is very difficult. Here are two known conjectures related to the question:

- (i) $(3/2)^n \bmod 1$ is uniformly distributed in $[0, 1]$.
- (ii) $(3/2)^n \bmod 1$ is dense in $[0, 1]$.

More related conjectures can be seen, for example, [14] by Strauch. Hence, we may ask similar questions on $((p+1)/p)^n \bmod 1$ for any $p \in \mathbb{N}$ as follows: For a positive integer p ,

- (iii) is $((p+1)/p)^n \bmod 1$ uniformly distributed in $[0, 1]$? and
 - (iv) is $((p+1)/p)^n \bmod 1$ dense in $[0, 1]$?
- Or more wider, for p and q are in $\mathbb{R}^+ \setminus \{1\}$ with $\log_q p \notin \mathbb{Q}$,
- (v) is $((p+q)/p)^n \bmod 1$ uniformly distributed in $[0, 1]$? and
 - (vi) is $((p+q)/p)^n \bmod 1$ dense in $[0, 1]$?

Finally we give a few more applications in the continuous solutions of functional equations.

Example 3.5 We now consider two extension of Cauchy equation shown in Example 2.1. First is the Jensen's equation

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}. \quad (3.2)$$

Let $g(x) = f(x) - f(0)$. Then $g(0) = 0$ and

$$g\left(\frac{x+y}{2}\right) = \frac{g(x) + g(y)}{2}. \quad (3.3)$$

It is easy to see (3.3) has a continuous solution $g(x) = ax$ for some constant a . Setting $y = 0$ in (3.3), we obtain $g(x/2) = g(x)/2$, which implies $g(2^n x) = 2^n g(x)$ for all $n \in \mathbb{Z}$ by using the same argument shown in Example 2.1. Similarly, setting $y = 2x$ in (3.3) yields $g(3x/2) = (3/2)g(x)$. Thus, $g(2^n 3^m x) = 2^n 3^m g(x)$. Therefore, from Corollary 2.2, (3.3) has a unique continuous solution $g(x) = ax$ for some $a \in \mathbb{R}$ and (3.2) has a unique continuous solution $f(x) = ax + b$ for some $a, b \in \mathbb{R}$.

The second equation is

$$f(ax + by + c) = af(x) + bf(y) + d \quad (3.4)$$

for constants $a, b, c, d \in \mathbb{R}$ and $a, b \neq 0$. Let $x = (s - c)/a$ and $y = t/b$. Then

$$f(s + t) = af\left(\frac{s - c}{a}\right) + bf\left(\frac{t}{b}\right) + d,$$

which yields

$$f(s) = af\left(\frac{s - c}{a}\right) + bf(0) + d,$$

$$f(t) = af\left(-\frac{c}{a}\right) + bf\left(\frac{t}{b}\right) + d,$$

and

$$f(0) = af\left(-\frac{c}{a}\right) + bf(0) + d.$$

Thus, we obtain

$$f(s + t) - f(s) - f(t) + f(0) = 0.$$

Let $g(s) = f(s) + f(0)$. Then the above equation gives the Cauchy equation $g(s + t) = g(s) + g(t)$. Thus, (3.4) has a unique solution $f(x) = mx + n$ for some constant $m, n \in \mathbb{R}$.

Example 3.5. To find all non-trivial continuous function $f : \mathbb{R} \mapsto \mathbb{R}$ of the following extension of Cauchy’s exponential equation (see Example 2.2),

$$f(\sqrt{x^2 + y^2}) = f(x)f(y) \tag{3.5}$$

for all $x, y \in \mathbb{R}$, we assume there exists a point x_0 such that $f(x_0) \neq 0$. Thus,

$$f(x)f(x_0) = f\left(\sqrt{x^2 + x_0^2}\right) = f(-x)f(x_0),$$

which implies f is an even function. Let $g(x) = f(\sqrt{x})$ for $x \geq 0$. Then $g(x + y) = g(x)g(y)$ for $x, y \geq 0$. Example 2.2 shows that $g(x) = b^x$. Therefore, $f(x) = b^{x^2}$. Since $b = g(1) = f(1)$, we finally obtain $f(x) = (f(1))^{x^2}$, which is the unique continuous solution of (3.5).

Acknowledgments. The authors would like to thank the editors for their helpful suggestions.

REFERENCES

- [1] J. Aczel and C. Alsina: *Trisection of angles, classical curves, and functional equations*, Math. Magazine, **71**(1998), No. 3, 182-189.
- [2] C. Alsina and R. Nelsen: *On Candido’s identity*, Math. Magazine, **80**(2007), No. 3, 226-228.
- [3] A. Berger: *Multi-dimensional dynamical systems and Benford’s law*, Discrete Contin. Dyn. Syst., **13**(2005), No. 1, 219-237.
- [4] A. Berger, L. A. Bunimovich and T. P. Hill: *One-dimensional dynamical systems and Benford’s law*, Trans. Amer. Math. Soc., **357**(2005), No. 1, 197-219.
- [5] P. Diaconis: *The distribution of leading digits and uniform distribution mod 1*, Ann. Probability, **5**(1977), No. 1, 72-81.

- [6] M. Einsiedler: *What is measure rigidity?*, Notices Amer. Math. Soc., **56**(2009), No. 5, 600-601.
- [7] G. H. Hardy and E. M. Wright: *An Introduction to the Theory of Numbers*, Fifth Edition, Oxford Science Publications, 1979.
- [8] T. X. He, Z. Sinkala and X. Zha: *If $F(x) = \int_x^{2x} f(t)dt$ is constant, must $f(t) = c/t$?*, College Math. J., **36**(2005), 199-204.
- [9] G. A. Heuer: *Rational numbers generated by two integers*, Am. Math. Mon., **78**(1971), No. 9, 996-997.
- [10] L. Kuipers and H. Niederreiter: *Uniform Distribution of Sequences*, Pure and Applied Mathematics. Wiley-Interscience, John Wiley & Sons, New York-London-Sydney, 1974.
- [11] L. Moser and N. Macon: *On the distribution of first digits of powers*, Scripta Math., **16**(1950), 290-292.
- [12] I. Niven: *Irrational Numbers*, MAA, John Wiley & Sons, Inc., 1956.
- [13] W. Rudin: *Principles of Mathematical Analysis*, Third Edition, McGraw-Hill, Inc., New York, 1976.
- [14] O. Strauch: *On distribution functions of $\xi(3/2)^n \bmod 1$* , Acta Arith., **81**(1997), No. 1, 25-35.

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