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AN IMPROVED CHEN-RICCI INEQUALITY FOR LEGENDRIAN SUBMANIFOLDS IN SASAKIAN SPACE FORMS

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ABSTRACT. B.Y. Chen [Arch. Math. (Basel), **74**(2000), 154-160] proved a geometrical inequality for Lagrangian submanifolds in complex space forms in terms of the Ricci curvature and the squared mean curvature. Recently, this Chen-Ricci inequality was improved in [Int. Electron. J. Geom., **2**(2009), 39-45]. On the other hand, the first author [J. Austral. Math. Soc., **72**(2002), 247-256] established a Chen-Ricci inequality for submanifolds, in particular C-totally real submanifolds, in Sasakian space forms. In this article, we improve the above inequality for Legendrian submanifolds in Sasakian space forms. We also investigate the equality case of the inequality.

1. PRELIMINARIES

Let $\widetilde{M}^{2m+1}$ be a $(2m+1)$-differentiable manifold. The triple (ϕ, ξ, η) on $\widetilde{M}^{2m+1}$ is called a (ϕ, ξ, η)-*structure* if it satisfies

$$\eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi,$$

where ϕ is an endomorphism of the tangent bundle, ξ is a vector field, η is a 1-form and I is the identity tensor.

We recall that $\phi\xi = 0$ and $\eta \circ \phi = 0$.

If the manifold $\widetilde{M}^{2m+1}$ with a (ϕ, ξ, η)-*structure* admits a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all vector fields X, Y, then $\widetilde{M}^{2m+1}$ has a (ϕ, ξ, η, g)-*almost contact metric structure*. For more details see [1].

A $(2m+1)$-dimensional Riemannian manifold $(\widetilde{M}^{2m+1}, g)$ is said to be a *Sasakian manifold* if it admits an almost contact metric structure, which is normal, i.e.,

$$(\widetilde{\nabla}_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \quad \widetilde{\nabla}_X \xi = -\phi X,$$

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for any vector fields X, Y on $T\widetilde{M}^{2m+1}$, where $\widetilde{\nabla}$ denotes the Riemannian connection with respect to g.

A plane section π in $T_p\widetilde{M}^{2m+1}$ is called a *ϕ-section* if it is spanned by X and ϕX, where X is a unit tangent vector orthogonal to ξ. The sectional curvature of a ϕ-section is called a *ϕ-sectional curvature.*

A Sasakian manifold with constant ϕ-sectional curvature c is said to be a *Sasakian space form* and is denoted by $\widetilde{M}^{2m+1}(c)$.

As examples of Sasakian space forms we have $\mathbb{R}^{2m+1}$ and S^{2m+1}, with standard Sasakian structures (see more details in [1], [12]).

The curvature tensor of a Sasakian space form $\widetilde{M}^{2m+1}(c)$ is given by

$$\begin{aligned}\widetilde{R}(X,Y)Z = \frac{c+3}{4}&\{g\left(Y,Z\right)X - g(X,Z)Y\}\\ &+\frac{c-1}{4}\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} + \eta\left(X\right)\eta\left(Z\right)Y\\ &-\eta\left(Y\right)\eta\left(Z\right)X + g\left(X,Z\right)\eta\left(Y\right)\xi - g\left(Y,Z\right)\eta\left(X\right)\xi, \qquad (1.1)\end{aligned}$$

for all vector fields X, Y, Z on $\widetilde{M}^{2m+1}(c)$.

A submanifold M^n of a Sasakian manifold $\widetilde{M}^{2m+1}$ normal to ξ is called a *C-totally real* submanifold.

On such a submanifold, ϕ maps any tangent vector to M^n at $p \in M^n$ into the normal space $T_p^{\perp}M^n$ (see [11]).

In particular, if $n = m$, i.e., M^n has the maximum dimension, then M^n is said to be a *Legendrian* submanifold.

We denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM$, $p \in M$, by h the second fundamental form and by R the Riemann curvature tensor of M.

Then the Gauss equation is given by:

$$\widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) - g(h(X,Z),h(Y,W)) + g(h(X,W),h(Y,Z)),$$

for any vectors X, Y, Z, W tangent to M^n.

Let $p \in M^n$ and $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of the tangent space T_pM. We denote by H the *mean curvature vector*, i.e.,

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$$

and by

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j))$$

the squared norm of the second fundamental form.

For a Legendrian submanifold M^n we may choose an orthonormal basis of $T_p^\perp M^n$ of the form $\{e_{n+1} = \phi e_1, \dots, e_{2n} = \phi e_n, e_{2n+1} = \xi\}$. One has

$$A_{\phi X} Y = A_{\phi Y} X, \quad \forall X, Y \in T_p M,$$

or equivalently,

$$h_{ij}^k = h_{ik}^j = h_{jk}^i, \quad \forall i, j, k = 1, \dots, n, \tag{1.2}$$

where A is the shape operator and

$$h_{ij}^k = g(h(e_i, e_j), \phi e_k), \quad i, j, k = 1, \dots, n.$$

Definition 1.1. A Legendrian *H-umbilical submanifold* M^n of a Sasakian manifold $\widetilde{M}^{2n+1}$ is a Legendrian submanifold for which the second fundamental form takes the following forms:

$$h(e_1, e_1) = \lambda \phi e_1, \quad h(e_2, e_2) = \dots = h(e_n, e_n) = \mu \phi e_1,$$
$$h(e_1, e_j) = \mu \phi e_j, \quad h(e_j, e_k) = 0, \;\; 2 \le j \ne k \le n.$$

2. RICCI CURVATURE OF SUBMANIFOLDS

In [2], B.Y. Chen established a sharp relationship between the Ricci curvature (an intrinsic invariant) and the squared mean curvature (the main extrinsic invariant) for any n-dimensional Riemannian submanifold of a real space form $\widetilde{M}(c)$ of constant sectional curvature c; namely,

$$Ric(X) \le (n-1)c + \frac{n^2}{4} \|H\|^2,$$

which is known as the Chen-Ricci inequality.

The same inequality holds for Lagrangian submanifolds M^n in a complex space form $\widetilde{M}^n(4c)$ as well (see [3]).

The first author proved in [10] a similar inequality for submanifolds of Sasakian space forms. Moreover K. Matsumoto and the first author [8] investigated the corresponding inequality for C-totally real submanifolds and characterized such submanifolds which satisfy identically the equality case.

The Chen-Ricci inequality was further improved by S. Deng [7] for Lagrangian submanifolds to the following.

Theorem 2.1. *Let M^n be a Lagrangian submanifold of dimension $n \ge 2$ in a complex space form $\widetilde{M}^n(4c)$ of constant holomorphic sectional curvature $4c$ and X a unit tangent vector in T_pM, $p \in M$. Then, we have:*

$$Ric(X) \le (n-1)\left(c + \frac{n}{4}\|H\|^2\right).$$

The equality sign holds for any unit tangent vector at p if and only if either:
(i) *p is a totally geodesic point, or*
(ii) *$n = 2$ and p is an H-umbilical point with $\lambda = 3\mu$.*

Moreover, Lagrangian submanifolds in complex space forms satisfying the equality case of the inequality were determined by S. Deng in [7]. More precisely, he proved the following.

Corollary 2.1. *Let M^n be a Lagrangian submanifold of real dimension $n \geq 2$ in a complex space form $\widetilde{M}^n(4c)$. If*

$$Ric(X) = (n-1)\left(c + \frac{n}{4}\|H\|^2\right),$$

for any unit tangent vector X of M^n, then either

(i) *M^n is a totally geodesic submanifold in $\widetilde{M}^n(4c)$ or*

(ii) *$n = 2$ and M^2 is a Lagrangian H-umbilical surface of $\widetilde{M}^2(4c)$ with $\lambda = 3\mu$.*

3. RICCI CURVATURE OF LEGENDRIAN SUBMANIFOLDS

K. Matsumoto and the first author [8] studied the Ricci tensor of a C-totally real submanifold in a Sasakian space form. They estimated the mean curvature of such submanifolds in terms of the Ricci tensor. Moreover they investigated the equality case of the inequality.

Denoting by $\mathcal{R}$ the maximum of the Ricci curvature of a C-totally real submanifold M^n in a Sasakian space form $\widetilde{M}^{2m+1}(c)$, i.e.,

$$\mathcal{R}(p) = \max\{Ric(u) | u \in T_pM^n, g(u,u) = 1\}, \quad p \in M^n,$$

it follows that

$$\mathcal{R} \leq \frac{1}{4}\{n^2\|H\|^2 + (n-1)(c+3)\}.$$

If a Legendrian submanifold M^n of a Sasakian space form $\widetilde{M}^{2n+1}(c)$ satisfies identically the equality case, then it is a minimal submanifold.

We want to point-out that the Riemannian invariant $\mathcal{R}$ is the Chen invariant $\delta(n-1)$ (see [4]).

In this section, we obtain a corresponding inequality to that in Theorem 2.1 for Legendrian submanifolds in Sasakian space forms, which improves the inequality in [8]. Moreover, we remark that the equality case of the improved inequality for the invariant $\mathcal{R}$ does not imply the minimality of the Legendrian submanifold.

We shall use the following two Lemmas from [7].

Lemma 3.1. *Let $f_1(x_1, x_2, \ldots, x_n)$ be a function in $\mathbb{R}^n$ given by the formula*

$$f_1(x_1, x_2, \ldots, x_n) = x_1 \sum_{j=2}^{n} x_j - \sum_{j=2}^{n} x_j^2.$$

If $x_1 + x_2 + \cdots + x_n = 2na$, we have

$$f_1(x_1, x_2, \ldots, x_n) \leq \frac{n-1}{4n}(x_1 + x_2 + \cdots + x_n)^2.$$

The equality sign holds if and only if $\frac{1}{n+1}x_1 = x_2 = \cdots = x_n = a$.

Lemma 3.2. *Let $f_2(x_1, x_2, \ldots, x_n)$ be a function in $\mathbb{R}^n$ given by the formula*

$$f_2(x_1, x_2, \ldots, x_n) = x_1 \sum_{j=2}^{n} x_j - x_1^2.$$

If $x_1 + x_2 + \cdots + x_n = 4a$, we have

$$f_2(x_1, x_2, \ldots, x_n) \leq \frac{1}{8}(x_1 + x_2 + \cdots + x_n)^2.$$

The equality sign holds if and only if $x_1 = a$, $x_2 + \cdots + x_n = 3a$.

The main result of this section is the following theorem.

Theorem 3.1. *Let M^n be an n-dimensional Legendrian submanifold in a Sasakian space form $\widetilde{M}^n(c)$ of constant ϕ-sectional curvature c. Then, for any unit tangent vector X to M^n, we have*

$$Ric(X) \leq \frac{n-1}{4}\left(c + 3 + n\|H\|^2\right). \tag{3.1}$$

The equality sign of (3.1) *holds identically if and only if either:*
(i) *M^n is totally geodesic, or*
(ii) *$n = 2$ and M^2 is an H-umbilical Legendrian surface with $\lambda = 3\mu$.*

Proof. We fix a point $p \in M^n$ and a unit vector $X \in T_pM^n$. We choose an orthonormal basis $\{e_1 = X, e_2, \ldots, e_n\} \subset T_pM^n$ and

$$\{e_{n+1} = \phi e_1, \ldots, e_{2n} = \phi e_n, e_{2n+1} = \xi\} \subset T_p^{\perp}M^n.$$

For $j = 2, \ldots, n$ the Gauss equation gives

$$\widetilde{R}(e_1, e_j, e_1, e_j) = R(e_1, e_j, e_1, e_j) - g(h(e_1, e_1), h(e_j, e_j)) + g(h(e_1, e_j), h(e_1, e_j)),$$

or equivalently,

$$\widetilde{R}(e_1, e_j, e_1, e_j) = R(e_1, e_j, e_1, e_j) - \sum_{r=1}^{n}(h_{11}^r h_{jj}^r - (h_{1j}^r)^2), \quad \forall j \in \overline{2, n}.$$

Since the Riemannian curvature tensor $\widetilde{R}$ on M^n is given by (see (1.1)

$$\widetilde{R}(X, Y, Z, W) = \frac{c+3}{4}\{g(X, Z)g(Y, W) - g(X, W)g(Y, Z)\},$$

we find

$$\widetilde{R}(e_1, e_j, e_1, e_j) = \frac{c+3}{4}. \tag{3.2}$$

By summing after $j = \overline{2, n}$, we get

$$(n-1)\frac{c+3}{4} = Ric(X) - \sum_{r=1}^{n}\sum_{j=2}^{n}\left[h_{11}^r h_{jj}^r - (h_{1j}^r)^2\right].$$

It follows that

$$Ric(X) - (n-1)\frac{c+3}{4} = \sum_{r=1}^{n}\sum_{j=2}^{n}\left[h_{11}^{r}h_{jj}^{r} - (h_{1j}^{r})^2\right]$$

$$\leq \sum_{r=1}^{n}\sum_{j=2}^{n} h_{11}^{r}h_{jj}^{r} - \sum_{j=2}^{n}(h_{1j}^{1})^2 - \sum_{j=2}^{n}(h_{1j}^{j})^2. \qquad (3.3)$$

Since M^n is a Legendrian submanifold, we have the relations (1.2) and

$$Ric(X) - (n-1)\frac{c+3}{4} \leq \sum_{r=1}^{n}\sum_{j=2}^{n} h_{11}^{r}h_{jj}^{r} - \sum_{j=2}^{n}(h_{11}^{j})^2 - \sum_{j=2}^{n}(h_{jj}^{1})^2. \qquad (3.4)$$

Now we put

$$f_1(h_{11}^1, h_{22}^1, \ldots, h_{nn}^1) = h_{11}^1\sum_{j=2}^{n} h_{jj}^1 - \sum_{j=2}^{n}(h_{jj}^1)^2$$

and

$$f_r(h_{11}^r, h_{22}^r, \ldots, h_{nn}^r) = h_{11}^r\sum_{j=2}^{n} h_{jj}^r - (h_{11}^r)^2, \quad \forall r \in \overline{2,n}.$$

Since $nH^1 = h_{11}^1 + h_{22}^1 + \cdots + h_{nn}^1$, we obtain by using Lemma 3.1 that

$$f_1(h_{11}^1, h_{22}^1, \ldots, h_{nn}^1) \leq \frac{n-1}{4n}(nH^1)^2 = \frac{n(n-1)}{4}(H^1)^2. \qquad (3.5)$$

By applying Lemma 3.2 for $2 \leq r \leq n$, we get

$$f_r(h_{11}^r, h_{22}^r, \ldots, h_{nn}^r) \leq \frac{1}{8}(nH^r)^2 = \frac{n^2}{8}(H^r)^2 \leq \frac{n(n-1)}{4}(H^r)^2. \qquad (3.6)$$

From (3.4), (3.5) and (3.6), we obtain

$$Ric(X) - (n-1)\frac{c+3}{4} \leq \frac{n(n-1)}{4}\sum_{r=1}^{n}(H^r)^2 = \frac{n(n-1)}{4}\left\|H\right\|^2.$$

Thus we have

$$Ric(X) \leq (n-1)\frac{c+3}{4} + \frac{n(n-1)}{4}\left\|H\right\|^2,$$

which implies (3.1).

Next we shall study the equality case. For $n \geq 3$, we choose ϕe_1 parallel to H and we have $H^r = 0$, for $r \geq 2$; from Lemma 3.2, we get

$$h_{1j}^1 = h_{11}^j = \frac{nH^j}{4} = 0, \quad \forall j \geq 2,$$

and

$$h_{jk}^1 = 0, \quad \forall j, k \geq 2, \ j \neq k.$$

From Lemma 3.1, we have $h_{11}^1 = (n+1)a$ and $h_{jj}^1 = a$, $\forall j \geq 2$, with $a = \dfrac{H^1}{2}$.

In (3.3) we computed $Ric(X) = Ric(e_1)$. Similarly, by computing $Ric(e_2)$ and using the equality, we get

$$h^r_{2j} = h^2_{jr} = 0, \ \ \forall r \neq 2, \ \ j \neq 2, \ \ r \neq j.$$

Then we obtain

$$\frac{h^2_{11}}{n+1} = h^2_{22} = \cdots = h^2_{nn} = \frac{H^2}{2} = 0.$$

The argument is also true for matrices $\left(h^r_{jk}\right)$ because the equality holds for all unit tangent vectors; so, $h^2_{2j} = h^j_{22} = \dfrac{H^j}{2} = 0, \ \ \forall j \geq 3.$

The matrix $\left(h^2_{jk}\right)$ (respectively the matrix $\left(h^r_{jk}\right)$) has only two possible nonzero entries $h^2_{12} = h^2_{21} = h^1_{22} = \dfrac{H^1}{2}$ (respectively $h^r_{1r} = h^r_{r1} = h^1_{rr} = \dfrac{H^1}{2}$ $\forall r \geq 3$). Now, by the Gauss equation we obtain

$$\widetilde{R}(e_2, e_j, e_2, e_j) = R(e_2, e_j, e_2, e_j) - \left(\frac{H^1}{2}\right)^2, \ \forall j \geq 3.$$

Similarly we get

$$\widetilde{R}(e_2, e_1, e_2, e_1) = R(e_2, e_1, e_2, e_1) - (n+1)\left(\frac{H^1}{2}\right)^2 + \left(\frac{H^1}{2}\right)^2.$$

After combining the last two relations, we find

$$Ric(e_2) - (n-1)\frac{c+3}{4} = 2(n-1)\left(\frac{H^1}{2}\right)^2.$$

On the other hand, the equality case of (3.1) implies that

$$Ric(e_2) - (n-1)\frac{c+3}{4} = \frac{n(n-1)}{4}\|H\|^2 = n(n-1)\left(\frac{H^1}{2}\right)^2.$$

Since $n \neq 1, 2$, by equating the last 2 equations we find $H^1 = 0$. Thus, $\left(h^r_{jk}\right)$ are all zero, i.e., M^n is a totally geodesic submanifold in $\widetilde{M}^{2n+1}(c)$.

Now, let us assume that $n = 2$. If M^2 is not totally geodesic, one has

$$h(e_1, e_1) = \lambda e_3, \ h(e_2, e_2) = \mu e_3, \ h(e_1, e_2) = \mu e_4,$$

with $\lambda = 3\mu = \frac{3H^1}{2}$, i.e., M is H-umbilical. □

Finally, we shall give an example of a Legendrian submanifold which admits a vector field satisfying the equality case of (3.1).

Let S^3 be the 3-dimensional sphere and X_1, X_2 and X_3 the vector fields which parallelize S^3. We consider the Riemannian metric g_1 on S^3 defined by:

$$g_1(X_1, X_1) = g_1(X_2, X_2) = \frac{1}{3}g_1(X_3, X_3) = 3;$$
$$g_1(X_i, X_j) = 0, \quad \text{for all } 1 \leq i \neq j \leq 3.$$

An orthonormal frame is $E_1 = \frac{\sqrt{3}}{3}X_1$, $E_2 = \frac{\sqrt{3}}{3}X_2$, $E_3 = \frac{1}{3}X_3$. A straighforward computation gives $Ric(E_1) = Ric(E_2) = -\frac{2}{3}$, $Ric(E_3) = 2$.

It is known (see [5]) that S^3 admits a non-totally geodesic minimal Legendrian immersion in S^7.

Clearly E_3 satisfies the equality case of the inequality (3.1).

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