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DUALITY OF MULTITIME VECTOR INTEGRAL PROGRAMMING WITH QUASIINVEXITY

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ABSTRACT. We consider a multitime vector variational problem (VVP) and a multitime vector fractional variational problem (VFP). In their recently published work [Balkan J. Geom. Appl., **16**(2011), No. 2, 90-101], Mititelu and Postolache established necessary efficiency conditions for the two problems. By means of these conditions and using the framework of (ρ, b)-quasiinvexity, in this work we introduce duality conditions of Mond-Weir-Zalmai type for the fractional problem (VFP) through weak, direct and converse duality theorems. Our approach allows us to obtain some duality conditions for (VVP) as special cases in this theory.

1. INTRODUCTION

Beginning with the research of Valentine [23], published in 1937, during the years, the variational problem with constraints known different steps in its development. In time, several authors have been interested in the study of (sufficient) efficiency conditions and duality for fractional vector programs in connection with generalized convexity. To quote some sources, see [6] by Jeyakumar and Mond, [7] by Khan and Hanson, [8] by Liang et al., [10] by Mititelu, [15] and [16] by Pitea et al., [19] by Reddy and Mukherjee, [20] by Singh and Hanson.

In 2007 Mititelu and Stancu-Minasian [12] considered the following vector (or multiobjective) fractional variational problem:

$$
\text{(MSP)} \quad \begin{cases} \text{Maximize} \left(\dfrac{\int_a^b f_1(t,x,\dot{x})dt}{\int_a^b k_1(t,x,\dot{x})dt}, \dots, \dfrac{\int_a^b f_p(t,x,\dot{x})dt}{\int_a^b k_p(t,x,\dot{x})dt} \right) \\ \text{subject to } x(a) = a_0,\ x(b) = b_0, \\ g(t,x,\dot{x}) \leqq 0,\ h(t,x,\dot{x}) = 0,\ \forall t \in I, \end{cases}
$$

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where $I = [a, b]$ is interval, $x = (x_1, \dots, x_n)\colon I \to \mathbb{R}^n$ is piecewise smooth function on I with $\dot{x} = \dfrac{dx}{dt}$ its derivative, $f_1, k_1, \dots, f_p, k_p\colon I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $g\colon I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ and $h\colon I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^q$ are functions of C^2-class.

Mititelu [9], Mititelu and Stancu-Minasian [12] established necessary efficiency (Pareto minimum) conditions for problem (MSP). Using generalized quasiinvex functions, they also developed a dual program theory stating weak, direct and converse duality theorems [12].

In 2006, Udrişte [21] studied a control variational problem within multitime framework, establishing some optimality conditions, that is he introduced a multitime maximum principle for this kind of problems. Meanwhile, Udrişte and Ţevy [22] established the multitime principle of maximum for a multitime variational problem. In 2009 Pitea, Udrişte and Mititelu [15], [16] considered the multitime vector variant of the problem (MSP) in geometrical language. Using curvilinear integrals they established necessary efficiency conditions and developed a duality theory for this problem. Recently, Mititelu and Postolache [11] studied the same objectives for multitime vector fractional and nonfractional variational problems on Riemannian manifolds, but using multiple integrals on a measurable set Ω. For a survey of recent developments in multiobjective optimization, we address the reader to [1], by Chinchuluun and Pardalos.

It is our purpose in the present paper to introduce necessary efficiency conditions for multitime vector fractional and nonfractional variational problems involving also multiple integrals on a measurable set Ω, but in the space $\mathbb{R}^n$. For these problems we develop some duality results through weak, direct and converse theorems.

Consider a measurable set Ω in $\mathbb{R}^m$ and the functions

$$\begin{aligned}
&x\colon \Omega \to \mathbb{R}^n, \quad X\colon \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm} \to \mathbb{R},\\
&f = (f_r)\colon \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm} \to \mathbb{R}^p, \quad k = (k_r)\colon \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm} \to \mathbb{R}^p,\\
&g = (g_\alpha)\colon \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm} \to \mathbb{R}^m, \quad h = (h_\beta)\colon \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm} \to \mathbb{R}^q,
\end{aligned}$$

where $m, q \in \mathbb{N}^*$, $r = \overline{1, n}$, $\alpha = \overline{1, m}$ and $\beta = \overline{1, q}$. Denote by $t = (t^1, \dots, t^m) = (t^\alpha)$ the current element of Ω and by $x = (x^1, \dots, x^n) = (x^k)$ the current element of $\mathbb{R}^n$. The arguments of X, f, k, g_α, h_β are $(t, x, x_\gamma) = (t, x(t), x_\gamma(t))$, where

$$x = x(t) = (x^1(t), \dots, x^n(t)) = (x^k(t)), \quad t \in \Omega$$

and

$$x_\gamma = x_\gamma(t) = \frac{\partial x}{\partial t^\gamma}(t), \quad \gamma = \overline{1, m}$$

is the first derivative of $x = x(t)$. We suppose that $X, f, k, g_\alpha, h_\beta \in C^2(\Omega)$.

In this paper, Euler-Lagrange equations of the form

$$\frac{\partial X}{\partial x^k} - \frac{\partial}{\partial t^1}\left(\frac{\partial X}{\partial x^k_{t^1}}\right) - \cdots - \frac{\partial}{\partial t^m}\left(\frac{\partial X}{\partial x^k_{t^m}}\right) = 0, \quad k = \overline{1, n}$$

are shortly written as

$$\frac{\partial X}{\partial x^k} - \frac{\partial}{\partial t^\gamma}\left(\frac{\partial X}{\partial x^k_\gamma}\right) = 0, \tag{1.1}$$

where we denoted $x^k_\gamma = \dfrac{\partial x^k}{\partial t^\gamma}$ and

$$\begin{aligned}\frac{\partial}{\partial t^\gamma}\left(\frac{\partial X}{\partial x^k_\gamma}\right) &= \frac{\partial^2 X}{\partial t^\gamma \partial x^k_\gamma} + \frac{\partial^2 X}{\partial x^i \partial x^k_\gamma}\frac{\partial x^i}{\partial t^\gamma} + \frac{\partial^2 X}{\partial x^i_s \partial x^k_\gamma}\frac{\partial x^i_s}{\partial t^\gamma}\\ &= \frac{\partial^2 X}{\partial t^\gamma \partial x^k_\gamma} + \frac{\partial^2 X}{\partial x^i \partial x^k_\gamma}\frac{\partial x^i}{\partial t^\gamma} + \frac{\partial^2 X}{\partial x^i_s \partial x^k_\gamma}\frac{\partial^2 x^i}{\partial t^s \partial t^\gamma}.\end{aligned}$$

In the following, we use the Euler-Ostrogradsky equations with the vector form given by (1.1), namely

$$\frac{\partial X}{\partial x} - \frac{\partial}{\partial t^\gamma}\left(\frac{\partial X}{\partial x_\gamma}\right) = 0.$$

We define the set $\Phi = \{x \colon \Omega \to M \mid x \text{ is piecewise smooth on } \Omega\}$, where Ω is a normed space with $\|x\| = \|x\|_\infty + \sum_{k=1}^{n} \|\dot{x}^k\|_\infty$.

Throughout the paper, for two vectors $v = (v_1, \dots, v_n)$ and $w = (w_1, \dots, w_n)$ the relations of the form $v = w$, $v < w$, $v \leqq w$, $v \leq w$ are defined as follows

$$\begin{aligned}&v = w \Leftrightarrow v_i = w_i,\ i = \overline{1,n}; \quad v < w \Leftrightarrow v_i < w_i,\ i = \overline{1,n};\\ &v \leqq w \Leftrightarrow v_i \leqq w_i,\ i = \overline{1,n}; \quad v \leq w \Leftrightarrow v \leqq w \text{ and } v \neq w.\end{aligned}$$

Important note. To simplify the presentation of this work, in our subsequent theory, we shall set

$$\pi_x(t) = (t, x(t), x_\gamma(t)), \quad \pi_{x^0}(t) = (t, x^0(t), x^0_\gamma(t)), \quad \pi_y(t) = (t, y(t), y_\gamma(t)).$$

On X, consider the functionals

$$F_r(x) = \int_\Omega f_r(\pi_x(t))dv, \quad K_r(x) = \int_\Omega k_r(\pi_x(t))dv, \quad r = \overline{1,p},$$

and assume $K_r(x) \neq 0, \quad r = \overline{1,p}$.

In this paper, we establish various conditions of Mond-Weir duality type for the following multitime vector fractional variational problem:

$$\text{(VFP)} \quad \begin{cases} \text{Maximize } \left(\dfrac{F_1(x)}{K_1(x)}, \dots, \dfrac{F_p(x)}{K_p(x)}\right) \\ \text{subject to } g_\alpha(\pi_x(t)) = 0,\ h_\beta(\pi_x(t)) \leqq 0, \\ \ x(t)\big|_{\partial\Omega} = u(t) \text{ (given)}, \forall t \in \Omega, \end{cases}$$

where the fractions are well defined and $dv = dt_1 dt_2 \cdots dt_n$.

In particular, there are obtained duality conditions for the following multitime vector nonfractional variational problem

$$(\text{VVP}) \quad \begin{cases} \text{Maximize } I[x] = \displaystyle\int_\Omega f(\pi_x(t))dv \\ \text{subject to } g_\alpha(\pi_x(t)) = 0,\ h_\beta(\pi_x(t)) \leqq 0, \\ \ x\big|_{\partial\Omega} = u(t), \forall t \in \Omega, \end{cases}$$

The two problems, stated above, have the same domain

$$\mathcal{D} = \{x \in \Phi \,|\, g_\alpha(\pi_x(t)) = 0,\ h_\beta(\pi_x(t)) \leqq 0,\ x\big|_{\partial\Omega} = u(t),\ \forall t \in \Omega\}.$$

2. NECESSARY EFFICIENCY CONDITIONS FOR PROBLEMS (VVP) AND (VFP)

In their joint paper [11], Mititelu and Postolache established necessary efficiency conditions in the geometric framework of a Riemannian manifold for problems (VVP) and (VFP). In what follows, we shall present these conditions adapted to the study in the real space $\mathbb{R}^n$.

A. Efficiency for multitime vector variational problem (VVP). The vector functional

$$I[x] = \int_\Omega f(\pi_x(t))dv,$$

can be written componentwise as

$$I[x] = (F_1(x), \dots, F_p(x)).$$

Definition 2.1 ([3])**.** A point $x^* \in \mathcal{D}$ is said to be *efficient solution* (Pareto minimum) to (VVP) if there exist no $x \in \mathcal{D}$ such that $I[x] \leq I[x^*]$.

Now, according to [11], in the $\mathbb{R}^n$ case we have the following necessary efficiency conditions for (VVP).

Theorem 2.1 (Necessary efficiency for (VVP))**.** *Consider the vector multitime variational problem* (VVP) *and let* $x = x(t) \in \mathcal{D}$ *be an efficiency solution to* (VVP)*. Then there are vector functions* $\tau \in \mathbb{R}^p$, $\lambda(t) \in \mathbb{R}^m$ *and* $\mu(t) \in \mathbb{R}^q$, *all being piecewise smooth functions, which satisfy the conditions*

$$(\text{VFJ}) \quad \begin{cases} \tau^r \dfrac{\partial f_r}{\partial x^k} + \lambda^\alpha(t)\dfrac{\partial g_\alpha}{\partial x^k} + \mu^\beta(t)\dfrac{\partial h_\beta}{\partial x^k} - \\ \qquad - \dfrac{\partial}{\partial t^\gamma}\left(\tau^r \dfrac{\partial f_r}{\partial x^k_\gamma} + \lambda^\alpha(t)\dfrac{\partial g_\alpha}{\partial x^k_\gamma} + \mu^\beta(t)\dfrac{\partial h_\beta}{\partial x^k_\gamma}\right) = 0, \\ \mu^\beta(t) h_\beta(\pi_x(t)) = 0 \text{ (no summation)},\ \beta = \overline{1,q}, \\ \tau = (\tau^r) \geqq 0,\ (\mu^\beta(t)) \geqq 0,\ t \in \Omega,\ \gamma = \overline{1,m},\ k = \overline{1,n}. \end{cases}$$

Definition 2.2 ([9])**.** The efficient solution $x^0 \in \mathcal{D}$ is called *normal efficient solution* to (VVP) if conditions (VFJ) are satisfied with

$$\tau \geq 0, \quad e'\tau = 1, \quad e = (1, \ldots, 1) \in \mathbb{R}^p,$$

where $'$ is the transposition sign.

B. Efficiency for multitime vector fractional variational problem (VFP). In this section, we present the necessary efficiency conditions for (VFP) established in [11], also adaptated to the study in $\mathbb{R}^n$. Remark that the definition of efficient solution for (VFP) is similar Definition 2.1.

Theorem 2.2 (Necessary efficiency for (VFP))**.** *Let $x = x(t) \in \mathcal{D}$ be a normal efficient solution to problem* (VFP)*. Then there exist vector $\tau = (\tau^r) \in \mathbb{R}^p$, piecewise smooth functions $\lambda = (\lambda^\alpha(t)) \in \mathbb{R}^{nm}$ and $\mu = (\mu^\beta(t)) \in \mathbb{R}^q$ that satisfy the conditions*

$$\text{(MFJ)} \quad \begin{cases} \tau^r \left[K_r(x) \dfrac{\partial f_r}{\partial x^k} - F_r(x) \dfrac{\partial k_r}{\partial x^k} \right] + \lambda^\alpha(t) \dfrac{\partial g_\alpha}{\partial x^k} + \mu^\beta(t) \dfrac{\partial h_\beta}{\partial x^k} - \\ \quad - \dfrac{\partial}{\partial t^\gamma} \left\{ \tau^r \left[K_r(x) \dfrac{\partial f_r}{\partial x^k_\gamma} - F_r(x) \dfrac{\partial k_r}{\partial x^k_\gamma} \right] + \lambda^\alpha(t) \dfrac{\partial g_\alpha}{\partial x^k_\gamma} + \mu^\beta(t) \dfrac{\partial h_\beta}{\partial x^k_\gamma} \right\} = 0, \\ \mu^\beta(t) h_\beta(\pi_x(t)) = 0 \text{ no summation, } \beta = \overline{1, q}, \\ \tau \geqq 0, \ e'\tau = 1, \ (\mu^\beta(t)) \geqq 0, \ t \in \Omega, \ \gamma = \overline{1, m}, \ k = \overline{1, n}. \end{cases}$$

The definition of normal efficient solution for (VFP) is similar Definition 2.2.

Let ρ be a real number, and $b \colon \Phi \times \Phi \to [0, \infty)$ a nonnegative functional. On Φ, consider the functional

$$\mathcal{X}(x) = \int_\Omega X(\pi_x(t)) dv.$$

Definition 2.3 ([21])**.** The functional $\mathcal{X}$ is said to be (strictly) (ρ, b)-*quasiinvex* at x^0 if there exist vector functions $\eta(t) = (\eta_1(t), \ldots, \eta_n(t)) \in \mathbb{R}^n$ of C^1-class with $\eta(t)\big|_{\partial\Omega} = 0$ and $\theta(x, x^0) \in \mathbb{R}^n$ such that for any x $(x \neq x^0)$,

$$\begin{aligned} &\mathcal{X}(x) \leqq \mathcal{X}(x^0) \ \Rightarrow \\ &b(x, x^0) \int_\Omega \left\{ \eta_k \frac{\partial X}{\partial x^k}(\pi_{x^0}(t)) + \frac{\partial \eta_k}{\partial t^\gamma} \frac{\partial X}{\partial x^k_\gamma}(\pi_{x^0}(t)) \right\} dv (<) \leqq -\rho b(x, x^0) \|\theta(x, x^0)\|^2. \end{aligned}$$

We underline that the notion of quasiinvexity is used in an appropriate form in a recent work by Nahak and Mohapatra [14] for a study of some multiobjective programming problems.

Having in mind the background, introduced above, now we can state and prove our results on duality [13], [22].

3. MOND-WEIR-ZALMAI TYPE DUALITY FOR (VFP)

Relative to (VFP), we consider the next multitime vector fractional variational dual problem

$$
\text{(ZMFD)}\quad \begin{cases} \text{Maximize } \left(\dfrac{F_1(y)}{K_1(y)}, \dots, \dfrac{F_p(y)}{K_p(y)}\right) \\ \text{subject to } \tau^r \left[K_r(y)\dfrac{\partial f_r}{\partial x^k} - F_r(y)\dfrac{\partial k_r}{\partial x^k}\right] + \lambda^\alpha(t)\dfrac{\partial g_\alpha}{\partial x^k} + \mu^\beta(t)\dfrac{\partial h_\beta}{\partial x^k} \\ -\dfrac{\partial}{\partial t^\gamma}\left(\tau^r \left[K_r(x)\dfrac{\partial f_r}{\partial x^k_\gamma} - F_r(x)\dfrac{\partial k_r}{\partial x^k_\gamma}\right] + \lambda^\alpha(t)\dfrac{\partial g_\alpha}{\partial x^k_\gamma} + \mu^\beta(t)\dfrac{\partial h_\beta}{\partial x^k_\gamma}\right) = 0, \\ \lambda^\alpha(t) g_\alpha(\pi_y(t)) + \mu^\beta(t) h_\beta(\pi_y(t)) \geqq 0,\ i = \overline{1,p},\ \alpha = \overline{1,m},\ \beta = \overline{1,q}, \\ \tau = (\tau^r) \geqq 0,\ e'\tau = 1,\ (\mu^\beta(t)) \geqq 0,\ y \in \Phi,\ y(t)\big|_{\partial\Omega} = u(t),\ t \in \Omega. \end{cases}
$$

Denote by $\varpi(x)$ the value of problem (VFP) at $x \in \mathcal{D}$ and be $\delta(y, \lambda, \eta, \nu)$ the value of the dual (ZMFD) at $(y, \lambda, \eta, \nu) \in \Delta$, where Δ is the domain of (ZMFD). In what follows we develop a duality theory between (VFP) and (ZMFD).

Theorem 3.1 (Weak duality)**.** *Let x and (y, τ, λ, μ) be feasible points of problems* (VFP) *and* (ZMFD)*, respectively. Assume that there are satisfied the next conditions:*

a) *for each $r = \overline{1,p}$ we have $F_r(x) > 0$, $K_r(x) > 0$, $\forall x \in X$;*

b) *for each $r = \overline{1,p}$, $F_r(x)$ is (ρ'_r, b)-quasiinvex at y and $-K_r(x)$ is (ρ''_r, b)-quasiinvex at y, all in respect to η and θ;*

c) $\displaystyle\int_\Omega \lambda^\alpha(t) g_\alpha(\pi_x(t))dv$ *is (ρ''', b)-quasiinvex at y with respect to η and θ;*

d) $\displaystyle\int_\Omega \mu^\beta(t) h_\beta(\pi_x(t))dv$ *is monotonic (ρ^4, b)-quasiinvex at y with respect to η and θ;*

e) *one of the functions of* b) *and* c) *is strictly (ρ, b)-quasiinvex at y with respect to η and θ;*

f) $\tau^r[\rho'_r K_r(y) + \rho''_r F_r(y)] + \rho''' + \rho^4 \geqq 0.$

Then $\varpi(x) \leq \delta(y, \tau, \lambda, \mu)$ is false.

Proof. From the first assumption of b), it follows

$$
\begin{aligned} & F_r(x) \leqq F_r(y) \ \Rightarrow \\ & b(x,y)\int_\Omega \left\{\eta_k \frac{\partial f_r}{\partial y^k} + \frac{\partial \eta_k}{\partial t^\gamma}\frac{\partial f_r}{\partial y^k_\gamma}\right\} dv \leqq -\rho'_r b(x,y)\|\theta(x,y)\|^2, \end{aligned} \tag{3.1}
$$

while from the second assumption, we get

$$
\begin{aligned} & -K_r(x) \leqq -K_r(y) \Rightarrow \\ & b(x,y)\int_\Omega \left\{-\eta_k \frac{\partial k_r}{\partial y^k} - \frac{\partial \eta_k}{\partial t^\gamma}\frac{\partial k_r}{\partial y^k_\gamma}\right\} dv \leqq -\rho''_r b(x,y)\int_a^b \|\theta(x,y)\|^2. \end{aligned} \tag{3.2}
$$

Multiplying relation (3.1) by $K_r(y) > 0$ and (3.2) by $F_r(y) > 0$, and summing the obtained implications side by side we get

$$
\begin{aligned}
&F_r(x)K_r(y) - K_r(x)F_r(y) \leqq 0 \Rightarrow \\
&b(x,y)\int_\Omega \left\{ \eta_k \left[K_r(y)\frac{\partial f_r}{\partial y^k} - F_r(y)\frac{\partial k_r}{\partial y^k} \right] + \frac{\partial \eta_k}{\partial t^\gamma}\left[K_r(y)\frac{\partial f_r}{\partial y^k_\gamma} - F_r(y)\frac{\partial k_r}{\partial y^k_\gamma} \right] \right\} dv \\
&\qquad \leqq -[\rho'_r K_r(y) + \rho''_r F_r(y)] b(x,y)\|\theta(x,y)\|^2.
\end{aligned}
\tag{3.3}
$$

Multiplying (3.3) by τ^r, $r = \overline{1,p}$ $(\tau \geq 0)$ and by summing over $r = \overline{1,p}$, we obtain

$$
\begin{aligned}
&\tau^r[F_r(x)K_r(y) - K_r(x)F_r(y)] \leqq 0 \Rightarrow \\
&b(x,y)\int_a^b \left\{ \eta_k \tau^r \left[K_r(y)\frac{\partial f_r}{\partial y^k} - F_r(y)\frac{\partial k_r}{\partial y^k} \right] + \frac{\partial \eta_k}{\partial t^\gamma}\tau^r\left[K_r(y)\frac{\partial f_r}{\partial y^k_\gamma} - F_r(y)\frac{\partial k_r}{\partial y^k_\gamma} \right] \right\} dv \\
&\qquad \leqq -b(x,y)\|\theta(x,y)\|^2 \tau^r[\rho'_r K_r(y) + \rho''_r F_r(y)].
\end{aligned}
\tag{3.4}
$$

From c) and d) respectively we have

$$
\begin{aligned}
&\int_\Omega \lambda^\alpha(t) g_\alpha(\pi_x(t))dv \leqq \int_\Omega \lambda^\alpha(t) g_\alpha(\pi_y(t))dv \Rightarrow \\
&b(x,y)\int_\Omega \left\{ \eta_k \left[\lambda^\alpha(t)'\frac{\partial g_\alpha}{\partial y^k} \right] + \frac{\partial \eta_k}{\partial t^\gamma}\left[\lambda^\alpha(t)\frac{\partial g_\alpha}{\partial y^k_\gamma} \right] \right\} dv \leqq -\rho'' b(x,y)\|\theta\|^2;
\end{aligned}
\tag{3.5}
$$

and

$$
\begin{aligned}
&\int_\Omega \mu^\beta(t) h_\beta(\pi_x(t))dv \leqq \int_\Omega \mu^\beta(t) h_\beta(\pi_y(t))dv \Rightarrow \\
&b(x,y)\int_\Omega \left\{ \eta_k \left[\mu^\beta(t)\frac{\partial h_\beta}{\partial y^k} \right] + \frac{\partial \eta_k}{\partial t^\gamma}\left[\mu^\beta(t)\frac{\partial h_\beta}{\partial y^k_\gamma} \right] \right\} dv \leqq -\rho''' b(x,y)\|\theta\|^2.
\end{aligned}
\tag{3.6}
$$

Summing now side by side the double implications (3.4), (3.5) and (3.6), and taking into account e) it follows

$$
\begin{aligned}
&\tau^r[F_r(x)K_r(y) - K_r(x)F_r(y)] + \int_\Omega [\lambda^\alpha(t) g_\alpha(\pi_x(t)) + \mu^\beta(t) h_\beta(\pi_x(t))]dv \\
&\qquad - \int_\Omega [\lambda^\alpha(t) g_\alpha(\pi_y(t)) + \mu^\beta(t) y_\beta(\pi_y(t))]dv \leqq 0 \Rightarrow \\
&b(x,y)\int_\Omega \left\{ \eta_k \left\{ \tau^r \left[K_r(y)\frac{\partial f_r}{\partial y^k} - F_r(y)\frac{\partial k_r}{\partial y^k} \right] + \lambda^\alpha(t)'\frac{\partial g_\alpha}{\partial y^k} + \mu^\beta(t)\frac{\partial h_\beta}{\partial y^k} \right\} \right. \\
&\qquad \left. + \frac{\partial \eta_k}{\partial t^\gamma}\tau^r \left[K_r(y)\frac{\partial f_r}{\partial y^k_\gamma} - F_r(y)\frac{\partial k_r}{\partial y^k_\gamma} \right] + \lambda^\alpha(t)\frac{\partial g_\alpha}{\partial y^k_\gamma} + \mu^\beta(t)\frac{\partial h_\beta}{\partial y^k_\gamma} \right\} dv \\
&< -b(x,y)\|\theta(x,y)\|^2 \{\tau^r[\rho'_r K_r(y) + \rho''_r F_r(y)] + \rho''' + \rho^4\}.
\end{aligned}
\tag{3.7}
$$

From the second implication of (3.7), we obtain $b(x,y) > 0$.

Then the second implication of (3.7) shortly, becomes

$$\int_\Omega \left[\eta_s \frac{\partial V}{\partial y^s} + \frac{\partial \eta_s}{\partial t^\gamma}\frac{\partial V}{\partial y^s_\gamma}\right] dt < -\|\theta(x,y)\|^2\{\tau^r[\rho'_r K_r(y) + \rho''_r F_r(y)] + \rho''' + \rho^4\}, \quad (3.8)$$

with

$$V = \tau^r[K_r(y)f_r(\pi_y(t)) - F_r(y)k_r(\pi_y(t))] + \lambda^\alpha(t)g_\alpha(\pi_y(t)) + \mu^\beta(t)h_\beta(\pi_y(t)).$$

For each index j, we have

$$\frac{\partial \eta_k}{\partial t^\gamma}\frac{\partial V}{\partial y^k_\gamma} = \frac{\partial}{\partial t^\gamma}\left(\eta_k \frac{\partial V}{\partial y^k_\gamma}\right) - \eta_k \frac{\partial}{\partial t^\gamma}\left(\frac{\partial V}{\partial y^k_\gamma}\right),$$

and, by integration, we obtain

$$\int_\Omega \frac{\partial \eta_k}{\partial t_j}\frac{\partial V}{\partial y^k_\gamma} dv = \int_\Omega \frac{\partial}{\partial t^\gamma}\left(\eta_k \frac{\partial V}{\partial y^k_\gamma}\right) dv - \int_\Omega \eta_k \frac{\partial}{\partial t^\gamma}\left(\frac{\partial V}{\partial y^k_\gamma}\right) dv.$$

Using the gradient formula, we get

$$\int_\Omega \frac{\partial}{\partial t^\gamma}\left(\eta_k \frac{\partial V}{\partial y^k_\gamma}\right) dv = \int_{\partial\Omega} \left(\eta_k \frac{\partial V}{\partial y^k_\gamma}\right) \vec{n}(t) d\sigma = 0,$$

where $\vec{n}(t)$ is unit vector to surface $\partial\Omega$ at the current point, and $\eta_s(t)\big|_{\partial\Omega} = 0$.

Then relation (3.8) becomes

$$\int_\Omega \eta_k \left[\frac{\partial V}{\partial y^k} - \frac{\partial}{\partial t^\gamma}\left(\frac{\partial V}{\partial y^k_\gamma}\right)\right] dv < -\|\theta(x,y)\|^2\{\tau^r[\rho'_r K_r(y) + \rho''_r F_r(y)] + \rho''' + \rho^4\}. \quad (3.9)$$

Taking into account the first constraint of problem (ZMFD), we have

$$\frac{\partial V}{\partial y^k} - \frac{\partial}{\partial t^\gamma}\left(\frac{\partial V}{\partial y^k_\gamma}\right) = 0$$

and relation (3.9) becomes

$$0 < -\|\theta(x,y)\|^2\{\tau^r[\rho'_r K_r(y) + \rho''_r F_r(y)] + \rho''' + \rho^4\}.$$

According to hypothesis f), this inequality becomes $0 < 0$ which is false. Then, from (3.7) is true the next relation

$$\begin{aligned}\tau^r[F_r(x)K_r(y) - K_r(x)F_r(y)] + \int_\Omega [\lambda^\alpha(t)g_\alpha(\pi_x(t)) + \mu^\beta(t)h_\beta(\pi_x(t))]dv \\ - \int_\Omega [\lambda^\alpha(t)g_\alpha(\pi_y(t)) + \mu^\beta(t)h_\beta(\pi_y(t))]dv > 0.\end{aligned} \quad (3.10)$$

Taking into account relations (3.5) and (3.6), from (3.10) it follows

$$\tau^r[F_r(x)K_r(y) - K_r(x)F_r(y)] > 0,$$

that contradicts relation (3.3). Therefore, $\varpi(x) \leq \delta(y,\tau,\lambda,\mu)$ is false. □

Theorem 3.2 (Direct duality). *Let x^0 be a normal efficient solution of primal* (VFP) *and suppose satisfied the hypotheses of Theorem* 3.1. *Then there are scalar $\tau^0 \in \mathbb{R}^p$ and the piecewise smooth functions $\lambda^0 \colon \Omega \to \mathbb{R}^m$ and $\mu^0 \colon \Omega \to \mathbb{R}^q$ such that $(x^0, \tau^0, \lambda^0, \mu^0)$ is an efficient solution to dual* (ZMFD) *and $\varpi(x^0) = \delta(x^0, \tau^0, \lambda^0, \mu^0)$.*

Proof. Since x^0 is a normal efficient solution to problem (VFP), according to Theorem 2.2 there are vector $\tau^0 = (\tau^r)^0 \in \mathbb{R}^p$ and the piecewise smooth functions $\lambda^0 = (\lambda^\alpha)^0 \colon \Omega \to \mathbb{R}^{nm}$ and $\mu^0 = (\mu^\beta)^0 \colon \Omega \to \mathbb{R}^q$ that satisfy relations (MFJ). We get $\mu^\beta(t) h_\beta(\pi_{x^0}(t)) = 0$, $\beta = \overline{1,q}$. Also $(\lambda^\alpha)^0(t)' g_\alpha(\pi_{x^0}(t)) = 0$ is true for all i and α. It follows that $(x^0, \tau^0, \lambda^0, \mu^0) \in \Delta$ and $\varpi(x^0) = \delta(x^0, \tau^0, \lambda^0, \mu^0)$. □

Theorem 3.3 (Converse duality). *Let $(x^0, \tau^0, \lambda^0, \mu^0)$ be an efficient solution of dual* (MVFD) *and suppose satisfied the following conditions:*

i) *$\bar{x}$ is a normal efficient solution of primal* (VFP);

ii) *the hypotheses of Theorem* 3.1 *are satisfied for $(y, \tau, \lambda, \mu) = (x^0, \tau^0, \lambda^0, \mu^0)$.*

Then $\bar{x} = x^0$ and moreover, $\varpi(x^0) = \delta(x^0, \tau^0, \lambda^0, \mu^0)$.

Proof. On the contrary, suppose that $\bar{x} \neq x^0$ and we shall obtain a contradiction. Because $\bar{x}$ is a normal efficient solution to (VFP) then, according to Theorem 2.2 there are vector $\bar{\tau} \in \mathbb{R}^p$ and vector functions $\bar{\lambda} = (\bar{\lambda}^\alpha) \colon \Omega \to \mathbb{R}^{nm}$ and $\bar{\mu} = (\bar{\mu}^\beta) \colon \Omega \to \mathbb{R}^q$ that satisfy conditions (MFJ). We obtain

$$\bar{\lambda}^\alpha(t) g_\alpha(\pi_{\bar{x}}(t)) + \mu^\beta(t) h_\beta(\pi_{\bar{x}}(t)) = 0$$

and so, $(\bar{x}, \bar{\tau}, \bar{\lambda}, \bar{\mu}) \in \Delta$. Moreover, $\varpi(\bar{x}) = \delta(\bar{x}, \bar{\tau}, \bar{\lambda}, \bar{\mu})$. According to Theorem 3.1 relation $\varpi(\bar{x}) \leq \delta(x^0, \tau^0, \lambda^0, \mu^0)$ is false. It results that the relation

$$\delta(\bar{x}, \bar{\tau}, \bar{\lambda}, \bar{\mu}) \leq \delta(x^0, \tau^0, \lambda^0, \mu^0),$$

is false. Therefore, the maximal efficiency of $(x^0, \tau^0, \lambda^0, \mu^0)$ is contradicted. Then, it gets that the supposition $\bar{x} \neq x^0$, above made, is false. It follows $\bar{x} = x^0$ and also $\bar{\tau} = \tau^0$, $\bar{\lambda} = \lambda^0$, $\bar{\mu} = \mu^0$. Finally, we obtain $\varpi(\bar{x}) = \delta(x^0, \tau^0, \lambda^0, \mu^0)$. □

Corollary 3.1 (Weak duality). *Let $x \in \mathcal{D}$ and $(y, \tau, \lambda, \mu) \in \Delta$ be feasible points of problems* (VFP) *and* (ZMFD). *Assume that are satisfied the conditions:*

a) *for each $r = \overline{1,p}$, integral $\displaystyle\int_\Omega [K_r(y) f_r(\pi_x(t)) - F_r(y) k_r(\pi_x(t))] dv$ is (ρ'_r, b)-quasiinvex at y with respect to η and θ;*

b) *$\displaystyle\int_\Omega [\lambda^\alpha(t) g_\alpha(\pi_x(t)) + \mu^\beta(t) h_\beta(\pi_x(t))] dv$ is (ρ, b)-quasiinvex at y with respect to η and θ;*

c) *one of the functions of* a)-b) *is strictly (ρ, b)-quasiinvex at y with respect to η and θ;*

d) $\tau^r \rho'_r + \rho \geqq 0$.

Then $\varpi(x) \leq \delta(y, \tau, \lambda, \mu)$ is false.

Corollary 3.2 (Direct duality). *Let x^0 be a normal efficient solution of the primal* (VFP) *and suppose satisfied the hypotheses of Theorem* 3.1. *Then there are vector*

$\tau^0 \in \mathbb{R}^p$ and the piecewise smooth functions $\lambda^0 = (\lambda^\alpha)^0 \colon \Omega \to \mathbb{R}^{nm}$ and $\mu^0 = (\mu^\alpha)^0 \colon \Omega \to \mathbb{R}^q$ such that $(x^0, \tau^0, \lambda^0, \mu^0)$ is an efficient solution of the dual (ZMFD) *and $\varpi(x^0) = \delta(x^0, \tau^0, \lambda^0, \mu^0)$.*

Corollary 3.3 (Converse duality)**.** *Let $(x^0, \tau^0, \lambda^0, \mu^0) \in \Delta$ be a efficient solution to the dual and* (ZMFD) *and assume satisfied the next conditions:*
i) *$\bar{x}$ is a normal efficient solution of the primal* (VFP)*;*
ii) *the hypotheses of Theorem* 3.1 *are satisfied with $(y, \tau, \lambda, \mu) = (x^0, \tau^0, \lambda^0, \mu^0)$.*
Then $\bar{x} = x^0$ and $\varpi(x^0) = \delta(x^0, \tau^0, \lambda^0, \mu^0)$.

The theory in this section allows to consider some special cases of interest. These ones make the object of the following two sections.

4. CASE STUDY 1: MOND-WEIR DUALITY

In this section, we consider $K_r(x) = 1, \quad r = \overline{1,p}$. Problem (VFP) becomes (VVP), and (ZMFD) becomes the following multitime vector variational problem :

$$
\text{(MWD)} \quad \begin{cases} \text{Maximize } \displaystyle\int_\Omega f(\pi_y(t))dv = (F_1(y), \dots, F_p(y)) \\ \text{subject to } \tau^r \dfrac{\partial f_r}{\partial y^k} + \lambda^\alpha(t)\dfrac{\partial g_\alpha}{\partial y^k} + \mu^\beta(t)\dfrac{\partial h_\beta}{\partial y^k} \\ \qquad\qquad -\dfrac{\partial}{\partial t^\gamma}\left(\tau^i \dfrac{\partial f_i}{\partial y^k_\gamma} + \lambda^\alpha(t)\dfrac{\partial g_\alpha}{\partial y^k_\gamma} + \mu^\beta(t)\dfrac{\partial h_\beta}{\partial y^k_\gamma}\right) = 0 \\ \lambda^\alpha(t)' g_\alpha(\pi_y(t)) + \mu^\beta(t) h_\beta(\pi_y(t)) \geqq 0 \ \text{[no summation]} \\ \tau \geq 0, \ e'\tau = 1, \ y(t)\big|_{\partial\Omega} = u(t), \ t \in \Omega. \end{cases}
$$

The domain of (VVP) is $\mathcal{D}$ and denote by Δ' the domain of (MWD). Denote by $\varpi(x)$ the value of problem (VVP) at $x \in \mathcal{D}$ and be $\delta(y, \tau, \lambda, \mu)$ the value of the dual (MWD) at $(y, \tau, \lambda, \mu) \in \Delta'$.

There exist the following duality theorems of Mond-Weir type between problems (VVP) and (MWD).

Theorem 4.1 (Weak duality)**.** *Let $x \in \mathcal{D}$ and $(y, \tau, \lambda, \mu) \in \Delta$ be feasible points of the problems* (VVP) *and* (MWD) *respectively. Assume the conditions hold:*

a) *for each $r = \overline{1,p}$, the integral $\displaystyle\int_\Omega f_r(\pi_x(t))dv$ is (ρ'_r, b)-quasiinvex at $x = y$ with respect to η and θ;*

b) *$\displaystyle\int_\Omega \lambda^\alpha(t) g_\alpha(\pi_x(t))dv$ [no summation] is (ρ''_α, b)-quasiinvex at y with respect to η and θ;*

c) *$\displaystyle\int_\Omega \mu^\beta(t)' h_\beta(\pi_x(t))dv$ [no summation] is monotonic (ρ'''_β, b)-quasiinvex at y with respect to η and θ;*

d) *one of the functions of* a)-c) *is strictly* (ρ, b)*-quasiinvex at* y *with respect to* η *and* θ*;*

e) $\tau^r \rho'_r + \sum_{\alpha=1}^{m} \rho''_\alpha + \sum_{\beta=1}^{q} \rho'''_\beta \geqq 0.$

Then $\varpi(x) \leq \delta(y, \tau, \lambda, \mu)$ *is false.*

Theorem 4.2 (Direct duality). *Let* x^0 *be a normal efficient solution of the primal* (VVP) *and suppose satisfied the hypotheses of Theorem* 4.1. *Then there are* $\tau^0 \in \mathbb{R}^p$ *and the piecewise smooth functions* $\lambda^0 \colon \Omega \to \mathbb{R}^m$ *and* $\mu^0 \colon \Omega \to \mathbb{R}^q$ *such that* $(x^0, \tau^0, \lambda^0, \mu^0)$ *is an efficient solution of the dual problem* (MWD). *Moreover,* $\varpi(x^0) = \delta(x^0, \tau^0, \lambda^0, \mu^0)$.

Theorem 4.3 (Converse duality). *Let* $(x^0, \tau^0, \lambda^0, \mu^0)$ *be an efficient solution of the dual* (MWD) *and suppose satisfied the following conditions:*

i) $\bar{x}$ *is a normal efficient solution of the primal* (VVP)*;*

ii) *the hypotheses of Theorem* 4.1 *are satisfied for* $(y, \tau, \lambda, \mu) = (x^0, \tau^0, \lambda^0, \mu^0)$.

Then $\bar{x} = x^0$ *and* $\varpi(x^0) = \delta(x^0, \tau^0, \lambda^0, \mu^0)$.

As we expected, Theorems 4.1-4.3 are particular cases of Theorems 3.1-3.3.

Corollary 4.1 (Weak duality). *Let* x *and* (y, τ, λ, μ) *be feasible points of problems* (VVP) *and* (MWD). *Assume that are satisfied the conditions:*

a) *for each* $r = \overline{1,p}$, *the integral* $\int_\Omega f_r(\pi_x(t))dv$ *is* (ρ'_r, b)*-quasiinvex at* y *with respect to* η *and* θ*;*

b) $\int_\Omega [\lambda^\alpha(t) g_\alpha(\pi_x(t)) + \mu^\beta(t) h_\beta(\pi_x(t))]dv$ *is* (ρ, b)*-quasiinvex at* y *with respect to* η *and* θ*;*

c) *one of the functions of* a)-b) *is strictly quasiinvex at* y *with respect to* η *and* θ*;*

d) $\tau^r \rho'_r + \rho \geqq 0$.

Then $\varpi(x) \leq \delta(y, \lambda, \mu, v)$ *is false.*

Corollary 4.2 (Direct duality). *Let* x^0 *be a normal efficient solution of primal* (VVP) *and suppose satisfied the hypotheses stated in Corollary* 4.1. *There are vector* $\tau^0 \in \mathbb{R}^p$ *and the piecewise smooth functions* $\lambda^0 = (\lambda^\alpha)^0 \colon \Omega \to \mathbb{R}^{nm}$ *and* $\mu^0 = (\mu^\beta)^0 \colon \Omega \to \mathbb{R}^q$ *such that* $(x^0, \tau^0, \lambda^0, \mu^0)$ *is an efficient solution of the dual* (MWD). *Moreover,* $\varpi(x^0) = \delta(x^0, \tau^0, \lambda^0, \mu^0)$.

Corollary 4.3 (Converse duality). *Let* $(x^0, \tau^0, \lambda^0, \mu^0)$ *be an efficient solution to the dual* (MWD) *and suppose satisfied the following conditions:*

i) $\bar{x}$ *is a normal efficient solution of primal* (VP)*;*

ii) *the hypotheses of Corollary* 4.1 *are satisfied for* $(y, \tau, \lambda, \mu) = (x^0, \tau^0, \lambda^0, \mu^0)$.

Then $\bar{x} = x^0$ *and* $\varpi(\bar{x}) = \delta(x^0, \tau^0, \lambda^0, \mu^0)$.

Corollaries 4.1-4.3 are particular cases of Corollaries 3.1-3.3.

5. CASE STUDY 2: WOLFE DUALITY

Dual problem of Wolfe type, associated to (VVP) is the next multitime vector variational problem of maximum

$$
\text{(WD)} \quad \begin{cases} \text{Maximize } \displaystyle\int_\Omega \{f(\pi_y(t)) + [\lambda^\alpha(t) g_\alpha(\pi_y(t)) + \mu^\beta(t) h_\beta(\pi_y(t))]e\} dv \\ \text{subject to } \tau^r \dfrac{\partial f_r}{\partial y^k} + \lambda^\alpha(t)\dfrac{\partial g_\alpha}{\partial y^k} + \mu^\beta(t)\dfrac{\partial h_\beta}{\partial y^k} - \\ \qquad - \dfrac{\partial}{\partial t^\gamma}\left(\tau^r \dfrac{\partial f_r}{\partial y^k_\gamma} + \lambda^\alpha(t)\dfrac{\partial g_\alpha}{\partial y^k_\gamma} + \mu^\beta(t)\dfrac{\partial h_\beta}{\partial y^k_\gamma}\right) = 0, \\ \mu^\beta(t) \geqq 0, \ \forall t \in \Omega, \ k = \overline{1,n}, \ j = \overline{1,m}, \ e'\tau = 1. \end{cases}
$$

Corollary 5.1 (Weak duality). *Let x and (y, τ, λ, μ) be feasible points of problems* (VVP) *and* (WD). *Assume that are satisfied the conditions:*

a) *for each $r = \overline{1,p}$, the integral $\int_\Omega f_r(\pi_x(t))dv$ is (ρ'_r, b)-quasiinvex at y with respect to η and θ;*

b) $\int_\Omega [\lambda^\alpha(t) g_\alpha(\pi_x(t)) + \mu^\beta(t) h_\beta(\pi_x(t))]dv$ *is (ρ, b)-quasiinvex at y with respect to η and θ;*

c) *one of the functions of* a)-b) *is strictly quasiinvex at y with respect to η and θ;*

d) $\tau^r \rho'_r + \rho \geqq 0$.

Then $\varpi(x) \leq \delta(y, \tau, \lambda, \mu)$ is false.

Corollary 5.2 (Direct duality). *Let x^0 be a normal efficient solution of primal* (VVP) *and suppose satisfied the hypotheses stated in Corollary* 5.1. *Then there are vector $\tau^0 \in \mathbb{R}^p$ and the piecewise smooth functions $\lambda^0 = (\lambda^\alpha)^0 \colon \Omega \to \mathbb{R}^{nm}$ and $\mu^0 = (\mu^\beta)^0 \colon \Omega \to \mathbb{R}^q$ such that $(x^0, \tau^0, \lambda^0, \mu^0)$ is an efficient solution of dual* (WD) *and moreover, $\varpi(x^0) = \delta(x^0, \tau^0, \lambda^0, \mu^0)$.*

Corollary 5.3 (Converse duality). *Let $(x^0, \tau^0, \lambda^0, \mu^0)$ be an efficient solution to the dual* (DWP) *and suppose satisfied the following conditions:*

i) *$\bar{x}$ is a normal efficient solution of the primal* (VP)*;*

ii) *the hypotheses of Corollary* 4.1 *are satisfied for $(y, \tau, \lambda, \mu) = (x^0, \tau^0, \lambda^0, \mu^0)$.*

Then $\bar{x} = x^0$ and moreover, $\varpi(\bar{x}) = \delta(x^0, \tau^0, \lambda^0, \mu^0)$.

As we easily can see, Corollaries 5.1-5.3, stated above, are particular cases of Corollaries 4.1-4.3.

6. CONCLUSION

In this paper, we considered a study of a multitime vector (or multiobjective) variational problem (VVP) and a multitime vector fractional variational problem (VFP). For the two vector variational problems, Mititelu and Postolache established necessary efficiency conditions in their joint research work [11]. It was the aim of the present paper to utilize these conditions, in order to establish new duality conditions of Mond-Weir-Zalmai type for the fractional problem (VFP) through weak, direct and converse duality theorems. As particular cases, we obtained some duality conditions for (VVP). The notion of (ρ, b)-quasiinvexity was the main ingredient in the development of our theory.

The present study, was motivated by the wide class of applications of fractional programming arising form applied areas including engineering design, game theory, stock cutting, portfolio selection, decision problems in management. For a review on theory and applications of multiobjective programming, we address the reader to the monograph by Jahn, [5].

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