

## ON THE FURTHER UNIFIED THEORY OF IDEAL GENERALIZED CLOSED SETS

JOSÉ SANABRIA, ENNIS ROSAS, CARLOS CARPINTERO AND MARGOT SALAS-BROWN

ABSTRACT. In this paper we introduce and study two new notions of sets called  $\mathcal{J}$ - $mng$ -closed sets and  $mn$ - $\mathcal{J}g$ -closed sets, which are defined on a nonempty set with two minimal structures and an ideal. These sets enables us to obtain unified properties for certain notions of generalized closed sets with respect to an ideal. Moreover, we establish a new separation property called the class  $mn$ - $T_{\mathcal{J}}$ , which generalizes the class  $mn$ - $T_{1/2}$  [17].

### 1. INTRODUCTION

After Levine's work [12], where he introduced the notions of generalized closed sets (briefly  $g$ -closed), many mathematician have been published their articles using notions related with generalization of closed sets (see [1]-[2], [4]-[8], [10], [12]-[14], [16], [18], [21], [23], [26], [29] and [32]). A particular attention is the article *Unified operation approach of generalized closed sets* by J. Dontchev, M. Ganster and T. Noiri [6], where they introduced and studied the notion of  $\mathcal{J}$ -generalized closed sets (briefly  $\mathcal{J}$ - $g$ -closed) using the notion of ideals on a topological space. Another mathematician using the notion of minimal structures have been studied some unified approach of generalized closed sets, as we mention some of them:  $m_X$ - $g$ -closed [27],  $mg^*$ -closed [19],  $mng$ -closed [17],  $m$ - $\mathcal{J}g$ -closed [22] and  $\mathcal{J}$ - $mng$ -closed [20] (or  $m\mathcal{J}g^*$ -closed in [31]). In this work, in order to unify modifications of  $g$ -closed sets, we define and study two new notions of sets called  $\mathcal{J}$ - $mng$ -closed sets and  $mn$ - $\mathcal{J}g$ -closed sets, which are defined on a nonempty set with two minimal structures and an ideal. Also, we introduce a new separation property called the class  $mn$ - $T_{\mathcal{J}}$ , which generalizes the class  $mn$ - $T_{1/2}$  [17].

### 2. PRELIMINARIES

On a topological space  $(X, \tau)$  the following variants of the notions of open sets are very well-known (see [1],[2], [4]-[23] and [26]-[32]). Semi-open sets,  $SO(X, \tau)$ , preopen sets,  $PO(X, \tau)$ ,  $\alpha$ -open sets,  $\alpha(X, \tau)$  or  $\tau^\alpha$ , semi-preopen sets or  $\beta$ -open sets,  $SPO(X, \tau)$  or  $\beta(X, \tau)$ ,  $b$ -open sets,  $BO(X, \tau)$ ,  $\delta$ -open sets,  $\delta(X, \tau)$  or  $\tau_\delta$ , regular

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Received: February 02, 2011. Revised: April 26, 2011.

2010 Mathematics Subject Classification: 54A05, 54A10, 54D10.

Key words and phrases:  $\mathcal{J}$ - $g$ -closed,  $m$ -structure,  $mng$ -closed,  $\mathcal{J}$ - $mng$ -closed,  $mn$ - $\mathcal{J}g$ -closed.

open sets,  $\text{RO}(X, \tau)$ . In the case that  $\alpha : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$  is an operator associated to a topology  $\tau$ , that is, for all  $U \in \tau$ ,  $U \subseteq \alpha(U)$ , we obtain the notion of  $\alpha$ -semi open sets:  $A \subset X$  is said to be  $\alpha$ -semi open set if there exists  $U \in \tau$  such that  $U \subseteq A \subseteq \alpha(U)$ . Now if  $\alpha, \beta$  are operators associated to a topology  $\tau$ ,  $A \subset X$  is said to be  $(\alpha, \beta)$ -semi open set (see [27]) if for each  $x \in A$ , there exists a  $\beta$ -semi open set  $V$  such that  $x \in V$  and  $\alpha(V) \subseteq A$ . The family of  $(\alpha, \beta)$ -semi open set is denoted by  $(\alpha, \beta)\text{-SO}(X, \tau)$ .

The complements of semi-open (resp. preopen,  $\alpha$ -open, semi-preopen,  $b$ -open,  $\delta$ -open,  $(\alpha, \beta)$ -semi open, regular open) sets are called semi-closed (resp. preclosed,  $\alpha$ -closed, semi-preclosed,  $b$ -closed,  $\delta$ -closed,  $(\alpha, \beta)$ -semi closed, regular closed) sets. For a subset  $A$  of  $(X, \tau)$ , the closure (resp. semi-closure, preclosure,  $\alpha$ -closure, semi-preclosure or  $\beta$ -closure,  $b$ -closure,  $\delta$ -closure,  $(\alpha, \beta)$ -semi closure), is denoted by  $\text{Cl}(A)$  (resp.  $\text{sCl}(A)$ ,  $\text{pCl}(A)$ ,  $\alpha\text{Cl}(A)$ ,  $\text{spCl}(A)$  or  $\beta\text{Cl}(A)$ ,  $\text{bCl}(A)$ ,  $\text{Cl}_\delta(A)$ ,  $(\alpha, \beta)\text{-sCl}(A)$ ), is defined as the intersection of all closed (resp. semi-closed, preclosed,  $\alpha$ -closed, semi-preclosed or  $\beta$ -closed,  $b$ -closed,  $\delta$ -closed,  $(\alpha, \beta)$ -semi closed) sets of  $X$  containing  $A$ .

The family of closed sets above mentioned are extended to a general class of sets called a generalized closed sets. The following definition give many well known concepts of generalized closed sets.

**Definition 2.1.** A subset  $A$  of a topological space  $(X, \tau)$  is said to be:

- (1)  $g$ -closed [12], if  $\text{Cl}(A) \subset U$  whenever  $A \subset U$  and  $U \in \tau$ .
- (2)  $gs$ -closed [1], if  $\text{sCl}(A) \subset U$  whenever  $A \subset U$  and  $U \in \tau$ .
- (3)  $gp$ -closed [18], if  $\text{pCl}(A) \subset U$  whenever  $A \subset U$  and  $U \in \tau$ .
- (4)  $\alpha g$ -closed [14], if  $\alpha\text{Cl}(A) \subset U$  whenever  $A \subset U$  and  $U \in \tau$ .
- (5)  $gsp$ -closed [4], if  $\text{spCl}(A) \subset U$  whenever  $A \subset U$  and  $U \in \tau$ .
- (6)  $\gamma g$ -closed [7], if  $\text{bCl}(A) \subset U$  whenever  $A \subset U$  and  $U \in \tau$ .
- (7)  $\delta g$ -closed [5], if  $\text{Cl}_\delta(A) \subset U$  whenever  $A \subset U$  and  $U \in \tau$ .
- (8)  $rg$ -closed [23], if  $\text{Cl}(A) \subset U$  whenever  $A \subset U$  and  $U \in \text{RO}(X, \tau)$ .
- (9)  $sg^*$ -closed [16], if  $\text{Cl}(A) \subset U$  whenever  $A \subset U$  and  $U \in \text{SO}(X, \tau)$ .
- (10)  $pg^*$ -closed [16], if  $\text{Cl}(A) \subset U$  whenever  $A \subset U$  and  $U \in \text{PO}(X, \tau)$ .
- (11)  $\alpha g^*$ -closed [16], if  $\text{Cl}(A) \subset U$  whenever  $A \subset U$  and  $U \in \alpha(X, \tau)$ .
- (12)  $\beta g^*$ -closed [16], if  $\text{Cl}(A) \subset U$  whenever  $A \subset U$  and  $U \in \text{SPO}(X, \tau)$ .
- (13)  $sg$ -closed [2], if  $\text{sCl}(A) \subset U$  whenever  $A \subset U$  and  $U \in \text{SO}(X, \tau)$ .
- (14)  $pg$ -closed [18], if  $\text{pCl}(A) \subset U$  whenever  $A \subset U$  and  $U \in \text{PO}(X, \tau)$ .
- (15)  $g\alpha$ -closed [13], if  $\alpha\text{Cl}(A) \subset U$  whenever  $A \subset U$  and  $U \in \alpha(X, \tau)$ .
- (16)  $pgs$ -closed [32], if  $\text{pCl}(A) \subset U$  whenever  $A \subset U$  and  $U \in \text{SO}(X, \tau)$ .
- (17)  $gpr$ -closed [8], if  $\text{pCl}(A) \subset U$  whenever  $A \subset U$  and  $U \in \text{RO}(X, \tau)$ .
- (18)  $\alpha gs$ -closed [26], if  $\alpha\text{Cl}(A) \subset U$  whenever  $A \subset U$  and  $U \in \text{SO}(X, \tau)$ .
- (19)  $(\alpha, \beta)\text{-sg}$ -closed [29], if  $(\alpha, \beta)\text{-sCl}(A) \subset U$  whenever  $A \subset U$  and  $U \in (\alpha, \beta)\text{-SO}(X, \tau)$ .

Recall that an ideal  $\mathcal{J}$  on a nonempty set  $X$  is a nonempty collection of subsets of  $X$  which satisfies the following two properties:

- (1)  $A \in \mathcal{J}$  and  $B \subset A$  implies  $B \in \mathcal{J}$ .
- (2)  $A \in \mathcal{J}$  and  $B \in \mathcal{J}$  implies  $A \cup B \in \mathcal{J}$ .

Let  $\mathcal{J}$  an ideal on  $X$  and  $\mathcal{G}$  a nonempty family of subsets of  $X$ . For a subset  $A$  of  $X$  we define the generalized local function of  $A$  with respect to  $\mathcal{J}$  and  $\mathcal{G}$  as follows:

$$A^*(\mathcal{J}, \mathcal{G}) = \{x \in X : U \cap A \notin \mathcal{J} \text{ for every } U \in \mathcal{G} \text{ such that } x \in U\}.$$

A topological space  $(X, \tau)$  with an ideal  $\mathcal{J}$  on  $X$  is called an ideal topological space and is denoted by  $(X, \tau, \mathcal{J})$ .

Using the concept of local function, we find in the literature the following class of generalized closed sets:

**Definition 2.2.** A subset  $A$  of an ideal topological space  $(X, \tau)$  is said to be:

- (1)  $\mathcal{J}$ - $g$ -closed [6], if  $A^*(\mathcal{J}, \tau) \subset U$  whenever  $A \subset U$  and  $U \in \tau$ .
- (2)  $g\mathcal{J}$ -closed [10], if  $A^*(\mathcal{J}, \text{SO}(X, \tau)) \subset U$  whenever  $A \subset U$  and  $U \in \tau$ .
- (3)  $\mathcal{J}$ - $sg$ -closed [20], if  $A^*(\mathcal{J}, \tau) \subset U$  whenever  $A \subset U$  and  $U \in \text{SO}(X, \tau)$ .
- (4)  $\mathcal{J}$ - $pg$ -closed [20], if  $A^*(\mathcal{J}, \tau) \subset U$  whenever  $A \subset U$  and  $U \in \text{PO}(X, \tau)$ .
- (5)  $\mathcal{J}$ - $\alpha g$ -closed [20], if  $A^*(\mathcal{J}, \tau) \subset U$  whenever  $A \subset U$  and  $U \in \alpha(X, \tau)$ .
- (6)  $\mathcal{J}$ - $\beta g$ -closed [20], if  $A^*(\mathcal{J}, \tau) \subset U$  whenever  $A \subset U$  and  $U \in \beta(X, \tau)$ .

### 3. MINIMAL STRUCTURES

**Definition 3.1.** Let  $X$  be a nonempty set and  $\mathcal{P}(X)$  the power set of  $X$ . A subfamily  $m_X$  of  $\mathcal{P}(X)$  is called a minimal structure (briefly  $m$ -structure) on  $X$  [24] if  $\emptyset \in m_X$  and  $X \in m_X$ .

A set  $X$  with an  $m$ -structure  $m_X$  (resp. two  $m$ -structures  $m_X$  and  $n_X$ ) is called an  $m$ -space (resp. bi  $m$ -space) and is denoted by  $(X, m_X)$  (resp.  $(X, m_X, n_X)$ ). Each member of  $m_X$  is said to be  $m_X$ -open and the complement of an  $m_X$ -open set is said to be  $m_X$ -closed. The family of all  $m_X$ -closed sets is denoted by  $m_X^c$ .

**Definition 3.2.** Let  $X$  be a nonempty set and  $m_X$  an  $m$ -structure on  $X$ . For a subset  $A$  of  $X$ , the  $m_X$ -closure of  $A$  and the  $m_X$ -interior of  $A$  are defined in [15] as follows:

- (1)  $m_X\text{-Cl}(A) = \cap\{F : A \subset F, X \setminus F \in m_X\}$ ,
- (2)  $m_X\text{-Int}(A) = \cup\{U : U \subset A, U \in m_X\}$ .

**Lemma 3.1.** [15] *Let  $X$  be a nonempty set and  $m_X$  a minimal structure on  $X$ . For subsets  $A$  and  $B$  of  $X$ , the following properties hold:*

- (1)  $m_X\text{-Cl}(X \setminus A) = X \setminus m_X\text{-Int}(A)$  and  $m_X\text{-Int}(X \setminus A) = X \setminus m_X\text{-Cl}(A)$ ,
- (2) *If  $(X \setminus A) \in m_X$ , then  $m_X\text{-Cl}(A) = A$  and if  $A \in m_X$ , then  $m_X\text{-Int}(A) = A$ ,*
- (3) *If  $A \subset B$ , then  $m_X\text{-Cl}(A) \subset m_X\text{-Cl}(B)$  and  $m_X\text{-Int}(A) \subset m_X\text{-Int}(B)$ ,*
- (4)  $m_X\text{-Cl}(m_X\text{-Cl}(A)) = m_X\text{-Cl}(A)$  and  $m_X\text{-Int}(m_X\text{-Int}(A)) = m_X\text{-Int}(A)$ .

**Lemma 3.2.** [24] *Let  $(X, m_X)$  be an  $m$ -space and  $A$  a subset of  $X$ . Then  $x \in m_X\text{-Cl}(A)$  if and only if  $U \cap A \neq \emptyset$  for every  $U \in m_X$  containing  $x$ .*

**Definition 3.3.** A minimal structure  $m_X$  on a nonempty set  $X$  is said to have property  $\mathcal{B}$  [15] if the union of any family of subsets belonging to  $m_X$  belongs to  $m_X$ .

**Remark 3.1.** Let  $(X, \tau)$  be a topological space. The families  $\text{SO}(X, \tau)$ ,  $\text{PO}(X, \tau)$ ,  $\alpha(X, \tau)$ ,  $\beta(X, \tau)$ ,  $\text{BO}(X, \tau)$ ,  $\tau_\delta$ ,  $(\alpha, \beta)\text{-SO}(X, \tau)$  and  $\text{RO}(X, \tau)$  are all minimal structures on  $X$  have property  $\mathcal{B}$ . It is well-known that  $\alpha(X, \tau)$  and  $\tau_\delta$  are topologies for  $X$  and the others are not topologies.

**Lemma 3.3.** [25] *Let  $X$  be a nonempty set and  $m_X$  a minimal structure on  $X$  satisfying property  $\mathcal{B}$ . For a subset  $A$  of  $X$ , the following properties hold:*

- (1)  $A \in m_X$  if and only if  $m_X\text{-Int}(A) = A$ ,
- (2)  $A$  is  $m_X$ -closed if and only if  $m_X\text{-Cl}(A) = A$ ,

Using the notions of minimal structure, many mathematician working in unify the notions given in Definition 2.1. This unification was obtained by T. Noiri in [17].

**Definition 3.4.** Let  $(X, m_X)$  be an  $m$ -space. A subset  $A$  of  $X$  is said to be  $m$ - $g$ -closed [28] if  $m_X\text{-Cl}(A) \subset U$  whenever  $A \subset U$  and  $U \in m_X$ .

**Definition 3.5.** Let  $(X, \tau)$  be a topological space and  $m_X$  an  $m$ -structure on  $X$ . A subset  $A$  of  $X$  is said to be  $mg^*$ -closed [19] if  $\text{Cl}(A) \subset U$ , whenever  $A \subset U$  and  $U \in m_X$ .

**Definition 3.6.** Let  $(X, m_X, n_X)$  be a bi  $m$ -space. A subset  $A$  of  $X$  is said to be  $mng$ -closed [17] if  $n_X\text{-Cl}(A) \subset U$  whenever  $A \subset U$  and  $U \in m_X$ .

**Definition 3.7.** Let  $(X, \tau, \mathcal{J})$  be an ideal topological space and  $m_X$  an  $m$ -structure on  $X$ . A subset  $A$  of  $X$  is said to be  $\mathcal{J}$ - $mg$ -closed [20], if  $A^*(\mathcal{J}, \tau) \subset U$ , whenever  $A \subset U$  and  $U \in m_X$ .

The above definition almost unify all the notions given in Definition 2.1. In [22], they introduced a new concept of local function using minimal structure and obtain a generalization of all concepts given in Definition 2.1.

If  $(X, m_X)$  (resp.  $(X, m_X, n_X)$ ) is a  $m$ -space (resp. a bi  $m$ -space) and  $\mathcal{J}$  is an ideal on  $X$ , then  $(X, m_X, \mathcal{J})$  (resp.  $(X, m_X, n_X, \mathcal{J})$ ) is called an ideal  $m$ -space (resp. ideal bi  $m$ -space).

**Definition 3.8.** Let  $(X, m_X, \mathcal{J})$  be an ideal  $m$ -space and  $A$  a subset of  $X$ . Then  $A^*(\mathcal{J}, m_X) = \{x \in X : A \cap U \notin \mathcal{J} \text{ for every } U \in m_X \text{ such that } x \in U\}$  is called the minimal local function [22] of  $A$  with respect to  $\mathcal{J}$  and  $m_X$ .

When there is no ambiguity, we will write  $A_m^*(\mathcal{J})$  or simply  $A_m^*$  for  $A^*(\mathcal{J}, m_X)$ . Observe that  $A_m^* = m_X\text{-Cl}(A)$  and  $A_m^*(\mathcal{P}(X)) = \emptyset$ . Furthermore,  $\emptyset_m^* = \emptyset$ .

**Theorem 3.1.** [22] *Let  $(X, m_X)$  be a  $m$ -space with  $\mathcal{J}, \mathcal{I}$  ideals on  $X$ , and let  $A, B$  subsets of  $X$ . Then*

- (1)  $A \subset B \Rightarrow A_m^* \subset B_m^*$ ,
- (2)  $A_m^* = m_X\text{-Cl}(A_m^*) \subset m_X\text{-Cl}(A)$ ,
- (3)  $(A_m^*)_m^* \subset A_m^*$ .

**Definition 3.9.** A subset  $A$  of an ideal  $m$ -space  $(X, m_X, \mathcal{J})$  is  $m_X\star$ -closed [22] if  $A_m^* \subset A$ . The complement of an  $m_X\star$ -closed is said to be  $m_X\star$ -open.

**Definition 3.10.** A subset  $A$  of an ideal  $m$ -space  $(X, m_X, \mathcal{J})$  is  $m$ - $\mathcal{J}g$ -closed [22] if  $A_m^* \subset U$  whenever  $A \subset U$  and  $U \in m_X$ .

#### 4. $\mathcal{J}$ - $mng$ -CLOSED SETS

**Definition 4.1.** Let  $(X, m_X, n_X)$  be a bi  $m$ -space and  $\mathcal{J}$  be an ideal on  $X$ . A subset  $A$  of  $X$  is said to be  $\mathcal{J}$ - $mng$ -closed if  $n_X\text{-Cl}(A) \setminus U \in \mathcal{J}$ , whenever  $A \subset U$  and  $U \in m_X$ . The complement of an  $\mathcal{J}$ - $mng$ -closed set is said to be  $\mathcal{J}$ - $mng$ -open.

Observe that in the above definition, we obtain the unification of all the concepts of generalized closed sets given in Definitions 2.1, 3.4, 3.5, 3.6 and 3.7, also it generalize the notions of  $mng$ -closed sets.

**Remark 4.1.** Let  $(X, \tau)$  be a topological space.

(1) If  $m_X = \tau$ ,  $n_X = \tau$  (resp.  $\text{SO}(X, \tau)$ ,  $\text{PO}(X, \tau)$ ,  $\alpha(X, \tau)$ ,  $\text{SPO}(X, \tau)$ ,  $\text{BO}(X, \tau)$ ,  $\tau_\delta$ ) and  $\mathcal{J} = \{\emptyset\}$ , then an  $\mathcal{J}$ - $mng$ -closed set is a  $g$ -closed (resp.  $gs$ -closed,  $gp$ -closed,  $\alpha g$ -closed,  $gsp$ -closed,  $\gamma g$ -closed,  $\delta g$ -closed) set.

(2) If  $m_X = n_X = \alpha(X, \tau)$  (resp.  $\text{SO}(X, \tau)$ ,  $\text{PO}(X, \tau)$ ) and  $\mathcal{J} = \{\emptyset\}$ , then an  $\mathcal{J}$ - $mng$ -closed set is a  $g\alpha$ -closed (resp.  $sg$ -closed,  $pg$ -closed) set.

(3) If  $m_X = \text{RO}(X, \tau)$  (resp.  $\text{SO}(X, \tau)$ ,  $\text{PO}(X, \tau)$ ,  $\alpha(X, \tau)$ ,  $\beta(X, \tau)$ ) and  $n_X = \tau$  and  $\mathcal{J} = \{\emptyset\}$ , then an  $\mathcal{J}$ - $mng$ -closed set is an  $rg$ -closed (resp.  $sg^*$ -closed,  $pg^*$ -closed,  $\alpha g^*$ -closed,  $\beta g^*$ -closed) set.

(4) If  $m_X = \text{SO}(X, \tau)$  (resp.  $\text{RO}(X, \tau)$ ) and  $n_X = \text{PO}(X, \tau)$  and  $\mathcal{J} = \{\emptyset\}$ , then an  $\mathcal{J}$ - $mng$ -closed set is a  $pgs$ -closed (resp.  $gpr$ -closed) set.

(5) If  $m_X = \text{SO}(X, \tau)$  and  $n_X = \alpha(X, \tau)$  and  $\mathcal{J} = \{\emptyset\}$ , then an  $\mathcal{J}$ - $mng$ -closed set is an  $\alpha gs$ -closed set.

(6) If  $\alpha$  and  $\beta$  are associated operators to the topology  $\tau$  on  $X$  and  $m_X = n_X = (\alpha, \beta)\text{-SO}(X, \tau)$  and  $\mathcal{J} = \{\emptyset\}$ , then an  $\mathcal{J}$ - $mng$ -closed set is an  $(\alpha, \beta)$ - $sg$ -closed set.

**Remark 4.2.** (1) Let  $(X, m_X, \mathcal{J})$  be an ideal  $m$ -space. If  $n_X = m_X$  and  $\mathcal{J} = \{\emptyset\}$ , then an  $\mathcal{J}$ - $mng$ -closed set is an  $m_X$ - $g$ -closed set.

(2) Let  $(X, \tau, \mathcal{J})$  be an ideal topological space and  $m_X$  an  $m$ -structure on  $X$ . If  $n_X = \tau$  and  $\mathcal{J} = \{\emptyset\}$ , then an  $\mathcal{J}$ - $mng$ -closed set is an  $mg^*$ -closed set.

(3) Let  $(X, m_X, n_X, \mathcal{J})$  be an ideal bi  $m$ -space. If  $\mathcal{J} = \{\emptyset\}$ , then an  $\mathcal{J}$ - $mng$ -closed set is an  $mng$ -closed set.

The following example shows that the union and the intersection of two  $\mathcal{J}$ - $mng$ -closed sets need not be an  $\mathcal{J}$ - $mng$ -closed set.

**Example 4.1.** Let  $X = \{a, b, c, d\}$ ,  $m_X = \{\emptyset, X, \{a\}, \{c, d\}, \{a, b, c\}\}$ ,  $n_X = \{\emptyset, X, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, d\}\}$  and  $\mathcal{J} = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$ . Then,  $n_X^c = \{\emptyset, X, \{c\}, \{a, b\}, \{a, d\}, \{b, c\}\}$ . Let  $A = \{a\}$ ,  $B = \{c\}$ ,  $C = \{a, c\}$ ,  $D = \{c, d\}$ ,  $E = \{b, c, d\}$  and  $F = \{a, c, d\}$ . Observe that,

$$\begin{aligned}
 n_X\text{-Cl}(A) &= n_X\text{-Cl}(\{a\}) = \{a\}, \\
 n_X\text{-Cl}(B) &= n_X\text{-Cl}(\{c\}) = \{c\}, \\
 n_X\text{-Cl}(C) &= n_X\text{-Cl}(\{a, c\}) = X, \\
 n_X\text{-Cl}(D) &= n_X\text{-Cl}(\{c, d\}) = X, \\
 n_X\text{-Cl}(E) &= n_X\text{-Cl}(\{b, c, d\}) = X, \\
 n_X\text{-Cl}(F) &= n_X\text{-Cl}(\{a, c, d\}) = X.
 \end{aligned}$$

Furthermore,  $n_X\text{-Cl}(A) \setminus U \in \mathcal{J}$  and  $n_X\text{-Cl}(B) \setminus V \in \mathcal{J}$  for every  $U, V \in m_X$  such that  $A \subset U$  and  $B \subset V$ . Therefore,  $A = \{a\}$  and  $B = \{c\}$  are  $\mathcal{J}$ -mng-closed sets. On the other hand,  $n_X\text{-Cl}(C) \setminus \{a, b, c\} = X \setminus \{a, b, c\} = \{d\} \notin \mathcal{J}$ . Thus,  $A \cup B = C = \{a, c\}$  is not  $\mathcal{J}$ -mng-closed. Finally,  $n_X\text{-Cl}(E) \setminus U \in \mathcal{J}$  and  $n_X\text{-Cl}(F) \setminus V \in \mathcal{J}$  for every  $U, V \in m_X$  such that  $E \subset U$  and  $F \subset V$ . Thus,  $E = \{b, c, d\}$  and  $F = \{a, c, d\}$  are  $\mathcal{J}$ -mng-closed sets and  $n_X\text{-Cl}(D) \setminus \{c, d\} = X \setminus \{c, d\} = \{a, b\} \notin \mathcal{J}$ . Therefore,  $E \cap F = D = \{c, d\}$  is not  $\mathcal{J}$ -mng-closed.

In the sequel, the ideal bi  $m$ -space  $(X, m_X, n_X, \mathcal{J})$  is simply denoted by  $X$ . We give some results related with  $\mathcal{J}$ -mng-closed sets.

**Proposition 4.1.** *Every mng-closed set is  $\mathcal{J}$ -mng-closed.*

*Proof.* Suppose that  $A$  is mng-closed and let  $U \in m_X$  such that  $A \subset U$ . Then,  $n_X\text{-Cl}(A) \subset U$  and hence  $n_X\text{-Cl}(A) \setminus U = \emptyset \in \mathcal{J}$  for each ideal  $\mathcal{J}$  on  $X$ . Therefore,  $A$  is  $\mathcal{J}$ -mng-closed. □

The following example shows that the converse of the above theorem is in general not true.

**Example 4.2.** Let  $X = \{a, b, c\}$ ,  $m_X = \{\emptyset, X, \{a\}, \{b\}\}$ ,  $n_X = \{\emptyset, X, \{a\}, \{c\}\}$  and  $\mathcal{J} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$ . Then,  $\{a\}$  is  $\mathcal{J}$ -mng-closed but not mng-closed in  $X$ , since  $n_X\text{-Cl}(\{a\}) = \{a, b\}$  is not contained in  $\{a\} \in m_X$ .

**Proposition 4.2.** *If  $A$  is  $\mathcal{J}$ -mng-closed and  $A \subset B \subset n_X\text{-Cl}(A)$ , then  $B$  is  $\mathcal{J}$ -mng-closed.*

*Proof.* Suppose that  $A$  is an  $\mathcal{J}$ -mng-closed set and  $A \subset B \subset n_X\text{-Cl}(A)$ . Let  $U \in m_X$  such that  $B \subset U$ . Then  $A \subset U$  and hence  $n_X\text{-Cl}(A) \setminus U \in \mathcal{J}$ . Since  $B \subset n_X\text{-Cl}(A)$ , we have  $n_X\text{-Cl}(B) \subset n_X\text{-Cl}(A)$  and  $n_X\text{-Cl}(B) \setminus U \subset n_X\text{-Cl}(A) \setminus U \in \mathcal{J}$ . Therefore,  $B$  is an  $\mathcal{J}$ -mng-closed set. □

**Proposition 4.3.** *A subset  $A$  of  $X$  is  $\mathcal{J}$ -mng-open if and only if  $F \setminus G \subset n_X\text{-Int}(A)$  for some  $G \in \mathcal{J}$ , whenever  $F \subset A$  and  $F$  is  $m_X$ -closed.*

*Proof.* Necessity: Suppose that  $A$  is  $\mathcal{J}$ -mng-open. Let  $F \subset A$  and  $F$  be  $m_X$ -closed. Then,  $X \setminus A \subset X \setminus F \in m_X$  and  $X \setminus A$  is  $\mathcal{J}$ -mng-closed. Therefore,  $n_X\text{-Cl}(X \setminus A) \setminus (X \setminus F) \in \mathcal{J}$  and hence,  $n_X\text{-Cl}(X \setminus A) \subset (X \setminus F) \cup G$  for some  $G \in \mathcal{J}$ . This implies that,  $X \setminus ((X \setminus F) \cup G) \subset X \setminus n_X\text{-Cl}(X \setminus A)$  and hence  $F \setminus G \subset n_X\text{-Int}(A)$ .

Sufficiency: Let  $X \setminus A \subset U$  and  $U \in m_X$ . Then,  $X \setminus U \subset A$  and  $X \setminus U$  is  $m_X$ -closed. By the hypothesis, we have  $(X \setminus U) \setminus G \subset n_X\text{-Int}(A) = X \setminus n_X\text{-Cl}(X \setminus A)$  for some  $G \in \mathcal{J}$ . This gives  $X \setminus (U \cup G) \subset X \setminus n_X\text{-Cl}(X \setminus A)$  and consequently

$n_X\text{-Cl}(X \setminus A) \subset U \cup G$  for some  $G \in \mathcal{J}$ . Therefore,  $n_X\text{-Cl}(X \setminus A) \setminus U \in \mathcal{J}$  and hence,  $X \setminus A$  is  $\mathcal{J}$ -mng-closed. □

**Definition 4.2.** Let  $(X, m_X)$  be an  $m$ -space and  $A$  a subset of  $X$ . The subset  $\Lambda_m(A)$  [3] is defined as follows:  $\Lambda_m(A) = \cap\{U : A \subset U \in m_X\}$ .

**Theorem 4.1.** A subset  $A$  of  $X$  is  $\mathcal{J}$ -mng-closed if and only if  $n_X\text{-Cl}(A) \setminus \Lambda_m(A) \in \mathcal{J}$ .

*Proof.* Necessity: Suppose that  $A$  is  $\mathcal{J}$ -mng-closed. Let  $U$  be any  $m_X$ -open set containing  $A$ . Then  $n_X\text{-Cl}(A) \setminus U \in \mathcal{J}$  and  $\Lambda_m(A) \subset U$ . Hence,  $n_X\text{-Cl}(A) \setminus \Lambda_m(A) \subset n_X\text{-Cl}(A) \setminus U \in \mathcal{J}$ .

Sufficiency: Suppose that  $n_X\text{-Cl}(A) \setminus \Lambda_m(A) \in \mathcal{J}$  and let  $V$  be any  $m_X$ -open set containing  $A$ . Then,

$$\begin{aligned} n_X\text{-Cl}(A) \setminus V &\subset \bigcup_{U \in m_X} \{n_X\text{-Cl}(A) \setminus U : A \subset U\} \\ &= n_X\text{-Cl}(A) \setminus \bigcap_{U \in m_X} \{U : A \subset U\} \\ &= n_X\text{-Cl}(A) \setminus \Lambda_m(A) \in \mathcal{J}. \end{aligned}$$

Thus,  $n_X\text{-Cl}(A) \setminus V \in \mathcal{J}$  and hence  $A$  is an  $\mathcal{J}$ -mng-closed set. □

### 5. $mn$ - $\mathcal{J}g$ -CLOSED SETS AND $mn$ - $T_{\mathcal{J}}$ -SPACES

The Definition 4.1 can not be described using the concept of minimal local functions [22]. In consequence, we define a new class of sets in order to unify all the given concepts in Definitions 2.1, 3.4, 3.5, 3.6, 3.7 and 3.10.

**Definition 5.1.** Let  $(X, m_X, n_X)$  be a bi  $m$ -space and  $\mathcal{J}$  be an ideal on  $X$ . A subset  $A$  of  $X$  is said to be  $mn$ - $\mathcal{J}g$ -closed if  $A_n^* \subset U$ , whenever  $A \subset U$  and  $U \in m_X$ . The complement of an  $mn$ - $\mathcal{J}g$ -closed set is said to be  $mn$ - $\mathcal{J}g$ -open.

**Remark 5.1.** Note that if  $\mathcal{J} = \{\emptyset\}$ , then

$$mn\text{-}\mathcal{J}g\text{-closed} \Leftrightarrow \mathcal{J}\text{-mng-closed} \Leftrightarrow mng\text{-closed}.$$

Thus, the Remarks 4.1 and 4.2 are also true if we replace  $\mathcal{J}$ -mng-closed by  $mn$ - $\mathcal{J}g$ -closed.

**Remark 5.2.** Let  $(X, \tau)$  be a topological space and  $\mathcal{J}$  be an ideal on  $X$ .

(1) If  $m_X = \tau$ ,  $n_X = \tau$  (resp.  $\text{SO}(X)$ ), then an  $mn$ - $\mathcal{J}g$ -closed set is an  $\mathcal{J}$ - $g$ -closed (resp.  $g\mathcal{J}$ -closed) set.

(2) If  $m_X = \text{SO}(X, \tau)$  (resp.  $\text{PO}(X, \tau)$ ,  $\alpha(X, \tau)$ ,  $\beta(X, \tau)$ ) and  $n_X = \tau$ , then an  $mn$ - $\mathcal{J}g$ -closed set is an  $\mathcal{J}$ - $sg$ -closed,  $\mathcal{J}$ - $pg$ -closed,  $\mathcal{J}$ - $\alpha g$ -closed,  $\mathcal{J}$ - $\beta g$ -closed) set.

**Remark 5.3.** (1) Let  $(X, \tau, \mathcal{J})$  be an ideal topological space and  $m_X$  an  $m$ -structure on  $X$ . If  $n_X = \tau$ , then an  $mn$ - $\mathcal{J}g$ -closed set is an  $\mathcal{J}$ - $mg$ -closed set.

(2) Let  $(X, m_X, \mathcal{J})$  be an ideal  $m$ -space. If  $n_X = m_X$ , then an  $mn$ - $\mathcal{J}g$ -closed set is an  $m$ - $\mathcal{J}g$ -closed set.

The following two examples shows that, in general, the notions of  $\mathcal{J}$ -*mng*-closed and *mn*- $\mathcal{J}g$ -closed are independent.

**Example 5.1.** Let  $X = \{a, b, c\}$ ,  $m_X = \{\emptyset, X, \{a\}, \{b\}\}$ ,  $n_X = \{\emptyset, X, \{a\}, \{c\}\}$  and  $\mathcal{J} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$ . Then,  $n_X^c = \{\emptyset, X, \{b, c\}, \{a, b\}\}$ . Let  $A = \{a\}$ . Observe that  $n_X\text{-Cl}(A) = n_X\text{-Cl}(\{a\}) = \{a, b\}$ ,  $A = \{a\} \subset \{a\} \in m_X$  and  $A = \{a\} \subset X \in m_X$ . Furthermore,

$$\begin{aligned} n_X\text{-Cl}(A) \setminus \{a\} &= \{a, b\} \setminus \{a\} = \{b\} \in \mathcal{J}, \\ n_X\text{-Cl}(A) \setminus X &= \{a, b\} \setminus X = \emptyset \in \mathcal{J}. \end{aligned}$$

Therefore,  $A = \{a\}$  is an  $\mathcal{J}$ -*mng*-closed set. On the other hand,  $A_n^* = \{a, b\}$  is not contained in  $\{a\} \in m_X$ . Thus,  $A = \{a\}$  is not *mn*- $\mathcal{J}g$ -closed.

**Example 5.2.** Let  $X = \{a, b, c, d\}$ ,  $m_X = \{\emptyset, X, \{c\}, \{a, b\}\}$ ,  $n_X = \{\emptyset, X, \{a\}, \{c\}\}$  and  $\mathcal{J} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$ . Then,  $n_X^c = \{\emptyset, X, \{b, c, d\}, \{a, b, d\}\}$ . Let  $A = \{c\}$ . Observe that  $n_X\text{-Cl}(A) = n_X\text{-Cl}(\{c\}) = \{b, c, d\}$ ,  $A = \{c\} \subset \{c\} \in m_X$ , but  $n_X\text{-Cl}(A) \setminus \{c\} = \{b, c, d\} \setminus \{c\} = \{b, d\} \notin \mathcal{J}$ . Therefore,  $A = \{c\}$  is not an  $\mathcal{J}$ -*mng*-closed set. On the other hand,  $A_n^* = \emptyset$  and so  $A_n^* \subset U$ , whenever  $A \subset U$  and  $U \in m_X$ . Therefore,  $A = \{c\}$  is an *mn*- $\mathcal{J}g$ -closed.

Now, we present some results related with *mn*- $\mathcal{J}g$ -closed sets.

**Proposition 5.1.** *Every mng-closed set is mn- $\mathcal{J}g$ -closed.*

*Proof.* Suppose that  $A$  is *mng*-closed and let  $U \in m_X$  such that  $A \subset U$ . Since  $A_n^* \subset n_X\text{-Cl}(A)$ , then  $A_n^* \subset U$ . Thus,  $A$  is also *mn*- $\mathcal{J}g$ -closed.  $\square$

The following example shows that the converse of the above theorem is not true.

**Example 5.3.** Consider the ideal bi *m*-space  $(X, m_X, n_X, \mathcal{J})$  given in Example 5.2. Then,  $A = \{c\}$  is an *mn*- $\mathcal{J}g$ -closed but not *mng*-closed in  $X$ , since  $n_X\text{-Cl}(A) = \{b, c, d\}$  is not contained in  $\{c\} \in m_X$ .

**Definition 5.2.** A minimal structure  $m_X$  on a nonempty set  $X$  is said to have property  $\mathcal{F}$  if any finite intersection of subsets belonging to  $m_X$  belongs to  $m_X$ .

**Definition 5.3.** Let  $(X, m_X)$  be an *m*-space. A family  $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$  of subsets of  $X$  is said to be  $m_X$ -locally finite if for each  $x \in X$ , there exists an  $m_X$ -open set  $U$  containing  $x$  and  $U$  intersects  $A_\lambda$  at most for finitely many  $\lambda$ .

Note that every finite family of subsets of  $(X, m_X)$  is  $m_X$ -locally finite.

**Proposition 5.2.** *Let  $(X, \tau)$  be a topological space and  $m_X$  an *m*-structure on  $X$  such that  $\tau \subset m_X$ . Then, every locally finite family of subsets of  $X$  is  $m_X$ -locally finite.*

The following example shows that the converse of the above proposition need not be true in general.

**Example 5.4.** Let  $X = \mathbb{R}$  be the real numbers and  $\tau = \tau_u$  the usual topology. Let  $\mathcal{A} = \{\{1/n\} : n = 1, 2, \dots\}$  and  $m_X = \text{SO}(X, \tau)$ . Then the family  $\mathcal{A}$  is  $m_X$ -locally finite since for every  $a \in X$  we have  $(a - 1, 0] \in m_X$  but  $\mathcal{A}$  is not locally finite.

**Lemma 5.1.** *Let  $(X, m_X, \mathcal{J})$  be an ideal  $m$ -space and  $m_X$  has property  $\mathcal{F}$ . If  $\{A_\lambda : \lambda \in \Lambda\}$  is an  $m_X$ -locally finite family of subsets of  $(X, m_X, \mathcal{J})$ , then*

$$\bigcup_{\lambda \in \Lambda} (A_\lambda)_m^* = \left( \bigcup_{\lambda \in \Lambda} A_\lambda \right)_m^*.$$

*Proof.* For every  $\lambda \in \Lambda$  we have  $A_\lambda \subset \bigcup_{\lambda \in \Lambda} A_\lambda$  and this implies  $(A_\lambda)_m^* \subset \left( \bigcup_{\lambda \in \Lambda} A_\lambda \right)_m^*$ .

Hence we have  $\bigcup_{\lambda \in \Lambda} (A_\lambda)_m^* \subset \left( \bigcup_{\lambda \in \Lambda} A_\lambda \right)_m^*$ .

Conversely, let  $x \in \left( \bigcup_{\lambda \in \Lambda} A_\lambda \right)_m^*$  and  $U$  an  $m_X$ -open set such that  $x \in U$ . Since

$\{A_\lambda : \lambda \in \Lambda\}$  is  $m_X$ -locally finite, there exists an  $m_X$ -open set  $V$  containing  $x$  and  $V$  intersects  $A_\lambda$  at most for finitely many  $\lambda$ , says  $A_{\lambda_1}, A_{\lambda_2}, \dots, A_{\lambda_k}$ . Note that

$U \cap V$  is an  $m_X$ -open set containing  $x$ . Since  $x \in \left( \bigcup_{\lambda \in \Lambda} A_\lambda \right)_m^*$ , then  $(U \cap V) \cap \bigcup_{\lambda \in \Lambda} A_\lambda \notin \mathcal{J}$  for every  $U \in m_X$  such that  $x \in U$ . It follows that

$\bigcup_{\lambda \in \Lambda} (U \cap V) \cap A_\lambda \notin \mathcal{J}$  for every  $U \in m_X$  such that  $x \in U$ . Therefore, there exists

$i=1$   $A_{\lambda_j} \in \{A_{\lambda_1}, A_{\lambda_2}, \dots, A_{\lambda_k}\}$  such that  $(U \cap V) \cap A_{\lambda_j} \notin \mathcal{J}$ , and hence  $U \cap A_{\lambda_j} \notin \mathcal{J}$ .

Thus, we obtain  $x \in (A_{\lambda_j})_m^*$  which implies  $x \in \bigcup_{i=1}^k (A_{\lambda_i})_m^* \subset \bigcup_{\lambda \in \Lambda} (A_\lambda)_m^*$ . This shows

that  $\left( \bigcup_{\lambda \in \Lambda} A_\lambda \right)_m^* \subset \bigcup_{\lambda \in \Lambda} (A_\lambda)_m^*$ . This completes the proof. □

**Theorem 5.1.** *Let  $(X, m_X, n_X, \mathcal{J})$  be an ideal bi  $m$ -space where  $n_X$  has the property  $\mathcal{F}$ . If  $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$  is an  $n_X$ -locally finite family of subsets of  $X$  and  $A_\lambda$  is  $mn$ - $\mathcal{J}g$ -closed for each  $\lambda \in \Lambda$ , then  $\bigcup_{\lambda \in \Lambda} A_\lambda$  is  $mn$ - $\mathcal{J}g$ -closed.*

*Proof.* Let  $U \in m_X$  such that  $\bigcup_{\lambda \in \Lambda} A_\lambda \subset U$ . Then,  $A_\lambda \subset U$  for each  $\lambda \in \Lambda$ . Since  $A_\lambda$

is  $mn$ - $\mathcal{J}g$ -closed for each  $\lambda \in \Lambda$ , we have  $(A_\lambda)_n^* \subset U$  and hence,  $\bigcup_{\lambda \in \Lambda} (A_\lambda)_n^* \subset U$ . By

Lemma 5.1, we have  $\left( \bigcup_{\lambda \in \Lambda} A_\lambda \right)_n^* \subset U$ . Therefore,  $\bigcup_{\lambda \in \Lambda} A_\lambda$  is  $mn$ - $\mathcal{J}g$ -closed. □

**Corollary 5.1.** *Let  $(X, m_X, n_X, \mathcal{J})$  be an ideal bi  $m$ -space where  $n_X$  has the property  $\mathcal{F}$ . If  $A$  and  $B$  are  $mn$ - $\mathcal{J}g$ -closed sets of  $X$ , then  $A \cup B$  is also an  $mn$ - $\mathcal{J}g$ -closed set.*

The following example shows that the condition that  $n_X$  has the property  $\mathcal{F}$  is necessary in the above corollary.

**Example 5.5.** Let  $X = \{a, b, c, d\}$ ,  $m_X = \{\emptyset, X, \{a, b\}, \{a, c, d\}\}$ ,  $n_X = \{\emptyset, X, \{a, b\}, \{b, c\}, \{c, d\}\}$  and  $\mathcal{J} = \{\emptyset, \{b\}, \{d\}, \{b, d\}\}$ . Then,  $A = \{a\}$  and  $B = \{c, d\}$  are  $mn$ - $\mathcal{J}g$ -closed sets but  $A \cup B = \{a, c, d\}$  is not  $mn$ - $\mathcal{J}g$ -closed in  $X$ , since  $(A \cup B)_n^* = X$  is not contained in  $\{a, c, d\} \in m_X$ .

**Definition 5.4.** Two minimal structures  $m_X$  and  $n_X$  on a nonempty set  $X$  are said to have property  $\mathcal{D}$  if  $M \in m_X$  and  $N \in n_X$  imply  $M \cap N \in n_X$ .

**Lemma 5.2.** Let  $(X, m_X, n_X, \mathcal{J})$  be an ideal bi  $m$ -space and  $m_X$  and  $n_X$  have property  $\mathcal{D}$ . If  $\{A_\lambda : \lambda \in \Lambda\}$  is an  $m_X$ -locally finite family of subsets of  $(X, m_X, n_X, \mathcal{J})$ , then

$$\bigcup_{\lambda \in \Lambda} (A_\lambda)_n^* = \left(\bigcup_{\lambda \in \Lambda} A_\lambda\right)_n^*.$$

*Proof.* Arguin as in the proof of Lemma 5.1. Just use the fact that  $m_X$  and  $n_X$  have property  $\mathcal{D}$  instead of  $m_X$  has property  $\mathcal{F}$ . □

**Theorem 5.2.** Let  $(X, m_X, n_X, \mathcal{J})$  be an ideal bi  $m$ -space where  $m_X$  and  $n_X$  have property  $\mathcal{D}$ . If  $\mathcal{A} = \{A_\lambda : \lambda \in \Lambda\}$  is an  $m_X$ -locally finite family of subsets of  $X$  and  $A_\lambda$  is  $mn$ - $\mathcal{J}g$ -closed for each  $\lambda \in \Lambda$ , then  $\bigcup_{\lambda \in \Lambda} A_\lambda$  is  $mn$ - $\mathcal{J}g$ -closed.

*Proof.* Follows from Lemma 5.2, using a similar argument as in the proof of Theorem 5.1. □

**Corollary 5.2.** Let  $(X, m_X, n_X, \mathcal{J})$  be an ideal bi  $m$ -space where  $m_X$  and  $n_X$  have property  $\mathcal{D}$ . If  $A$  and  $B$  are  $mn$ - $\mathcal{J}g$ -closed sets of  $X$ , then  $A \cup B$  is also an  $mn$ - $\mathcal{J}g$ -closed set.

**Proposition 5.3.** If  $A$  is  $mn$ - $\mathcal{J}g$ -closed and  $A \subset B \subset A_n^*$ , then  $B$  is  $mn$ - $\mathcal{J}g$ -closed.

*Proof.* Suppose that  $A$  is an  $mn$ - $\mathcal{J}g$ -closed set and  $A \subset B \subset A_n^*$ . Let  $U \in m_X$  such that  $B \subset U$ . Then  $A \subset U$  and hence  $A_m^* \subset U$ . Since  $B \subset A_n^*$ , we have  $B_n^* \subset (A_n^*)_n^* \subset A_m^* \subset U$ . Therefore,  $B$  is an  $mn$ - $\mathcal{J}g$ -closed set. □

**Theorem 5.3.** A subset  $A$  of  $X$  is  $mn$ - $\mathcal{J}g$ -closed if and only if  $A_n^* \subset \Lambda_m(A)$ .

*Proof.* Necessity: Suppose that  $A$  is  $mn$ - $\mathcal{J}g$ -closed. Let  $U$  be any  $m_X$ -open set containing  $A$ . Then  $A_n^* \subset U$  and hence, we obtain  $A_n^* \subset \Lambda_m(A)$ .

Sufficiency: Suppose that  $A_n^* \subset \Lambda_m(A)$  and let  $U$  be any  $m_X$ -open set containing  $A$ . Then,  $A_n^* \subset \Lambda_m(A) \subset U$  and hence  $A$  is an  $mn$ - $\mathcal{J}g$ -closed set. □

**Proposition 5.4.** If  $m_X$  has property  $\mathcal{B}$  and  $A$  is a subset of  $X$ , then the following properties are equivalent:

- (1)  $A$  is  $mn$ - $\mathcal{J}g$ -closed.
- (2)  $n_X\text{-Cl}(A_n^*) \subset U$  for every  $m_X$ -open set  $U$  containing  $A$ .
- (3)  $m_X\text{-Cl}(\{x\}) \cap A \neq \emptyset$  for every  $x \in n_X\text{-Cl}(A_n^*)$ .

*Proof.* (1)  $\Rightarrow$  (2). Suppose that  $A$  is  $mn$ - $\mathcal{J}g$ -closed. Let  $U \in m_X$  such that  $A \subset U$ . Then,  $A_n^* \subset U$  and so by Theorem 3.1,  $n_X\text{-Cl}(A_n^*) \subset U$  whenever  $A \subset U$  and  $U \in m_X$ .

(2)  $\Rightarrow$  (3). Let  $x \in n_X\text{-Cl}(A_n^*)$  and suppose that  $m_X\text{-Cl}(\{x\}) \cap A = \emptyset$ . Then,  $A \subset X \setminus m_X\text{-Cl}(\{x\})$  and  $X \setminus m_X\text{-Cl}(\{x\})$  is  $m_X$ -open, since that  $m_X$  has property  $\mathcal{B}$ . By (2), we have  $n_X\text{-Cl}(A_n^*) \subset X \setminus m_X\text{-Cl}(\{x\}) \subset X \setminus \{x\}$ . This is a contradiction. Therefore,  $m_X\text{-Cl}(\{x\}) \cap A \neq \emptyset$ .

(3)  $\Rightarrow$  (1). Suppose that  $A$  is not  $mn$ - $\mathcal{J}g$ -closed. Then, there exists an  $m_X$ -open set  $U$  such that  $A \subset U$  and  $A_n^*$  is not contained in  $U$ . Therefore, there exists a point  $x \in A_n^*$  such that  $x \notin U$ . Thus, we have  $\{x\} \cap U = \emptyset$  and hence,  $m_X\text{-Cl}(\{x\}) \cap U = \emptyset$ . Since  $A \subset U$ , we obtain  $m_X\text{-Cl}(\{x\}) \cap A = \emptyset$ . By Theorem 3.1, we have  $x \in A_n^* = n_X\text{-Cl}(A_n^*)$  and it follows that (3) does not hold. This complete the proof.  $\square$

The following example shows that the implication (2)  $\Rightarrow$  (3) of Theorem 5.4 is not satisfied when  $m_X$  does not have property  $\mathcal{B}$ .

**Example 5.6.** Let  $X = \{a, b, c, d\}$ ,  $m_X = \{\emptyset, X, \{a, b\}, \{b, c\}\}$ ,  $n_X = \{\emptyset, X, \{a\}, \{a, c\}, \{c, d\}\}$  and  $\mathcal{J} = \{\emptyset, \{b\}\}$ . Let  $A = \{a, c\}$ , then  $n_X\text{-Cl}(A_n^*) = A_n^* = X$  and  $X$  is the only  $m_X$ -open set containing  $A$ . Thus, (2) is satisfied. On the other hand,  $d \in n_X\text{-Cl}(A_n^*) = X$  and  $m_X\text{-Cl}(\{d\}) = \{d\}$ . It follows that  $m_X\text{-Cl}(\{d\}) \cap A = \emptyset$  and hence (3) is not satisfied.

**Definition 5.5.** Minimal structures  $m_X$  and  $n_X$  on a nonempty set  $X$  are said to have property  $\mathcal{C}$  [17] if  $M \in m_X$  and  $N \in n_X$  imply  $M \cup N \in m_X$ .

**Theorem 5.4.** *If a subset  $A$  of  $X$  is  $mn$ - $\mathcal{J}g$ -closed, then  $n_X\text{-Cl}(A_n^*) \setminus A = A_n^* \setminus A$  does not contain any nonempty  $m_X$ -closed. The converse holds if  $n_X$  has property  $\mathcal{B}$  and  $m_X$  and  $n_X$  have property  $\mathcal{C}$ .*

*Proof.* Suppose that  $A$  is  $mn$ - $\mathcal{J}g$ -closed. Assume that  $F \subset A_n^* \setminus A$  and  $F$  is  $m_X$ -closed. Observe that  $A \subset X \setminus F$  and  $X \setminus F$  is  $m_X$ -open. Then,  $A_n^* \subset X \setminus F$  and  $F \subset X \setminus A_n^*$ . Since  $F \subset A_n^*$ , we have  $F \subset (X \setminus A_n^*) \cap A_n^* = \emptyset$ . Thus,  $n_X\text{-Cl}(A_n^*) \setminus A = A_n^* \setminus A$  does not contain any nonempty  $m_X$ -closed.

Conversely, suppose that  $n_X$  has property  $\mathcal{B}$  and  $m_X$  and  $n_X$  has property  $\mathcal{C}$ . Let  $A \subset U$  and  $U \in m_X$ . Since  $n_X\text{-Cl}(A_n^*) = A_n^*$  and  $n_X$  has property  $\mathcal{B}$ , then  $A_n^*$  is  $n_X$ -closed. Note that  $A_n^* \cap (X \setminus U) \subset A_n^* \setminus A$  and  $(X \setminus U)$  is  $m_X$ -closed. Since  $m_X$  and  $n_X$  have property  $\mathcal{C}$ , then  $A_n^* \cap (X \setminus U)$  is an  $m_X$ -closed set contained in  $A_n^* \setminus A$ . By assumption,  $A_n^* \cap (X \setminus U) = \emptyset$  and hence  $A_n^* \subset U$ . Thus,  $A$  is  $mn$ - $\mathcal{J}g$ -closed.  $\square$

The following example shows that, in the theorem above, the conditions that  $n_X$  has property  $\mathcal{B}$  and  $m_X$  and  $n_X$  have property  $\mathcal{C}$  cannot be dropped.

**Example 5.7.** Consider the ideal bi  $m$ -space  $(X, m_X, n_X, \mathcal{J})$  given in Example 5.2. If  $B = \{a\}$ , then  $B_n^* = \{a, b, d\}$  and  $B = \{a\} \subset \{a, b\} \in m_X$  but  $B_n^* = \{a, b, d\}$  is not contained in  $\{a, b\}$ . Therefore,  $B = \{a\}$  is not an  $mn$ - $\mathcal{J}g$ -closed set. On the other hand,  $m_X^c = \{\emptyset, X, \{c, d\}, \{a, b, d\}\}$ . Thus,  $B_n^* \setminus B = \{b, d\}$  does not contain any nonempty  $m_X$ -closed.

**Proposition 5.5.** *A subset  $A$  of  $X$  is  $mn$ - $\mathcal{J}g$ -closed if and only if  $A_n^* \cap F = \emptyset$  whenever  $A \cap F = \emptyset$  and  $F$  is  $m_X$ -closed.*

*Proof.* Necessity: Suppose that  $A$  is a  $mn$ - $\mathcal{J}g$ -closed set. Let  $A \cap F = \emptyset$  and  $F$  be  $m_X$ -closed. Then  $A \subset X \setminus F$  and  $X \setminus F$  is  $m_X$ -open. Therefore,  $A_n^* \subset X \setminus F$  and  $A_n^* \cap F = \emptyset$ .

Sufficiency: Let  $A \subset U$  and  $U$  is  $m_X$ -open. Then,  $A \cap (X \setminus U) = \emptyset$  and  $X \setminus U$  is  $m_X$ -closed. By the hypothesis we have  $A_n^* \cap (X \setminus U) = \emptyset$  and hence,  $A_n^* \subset U$ . This shows that  $A$  is a  $mn$ - $\mathcal{J}g$ -closed set. □

**Definition 5.6.** An ideal bi  $m$ -space  $(X, m_X, n_X, \mathcal{J})$ , is said to be  $mn$ - $T_{\mathcal{J}}$  if every  $mn$ - $\mathcal{J}g$ -closed set of  $X$  is  $n_X$ -closed.

**Theorem 5.5.** *For an ideal bi  $m$ -space  $(X, m_X, n_X, \mathcal{J})$ , the following properties are equivalent:*

- (1)  $X$  is  $mn$ - $T_{\mathcal{J}}$ ,
- (2) For each  $x \in X$ ,  $\{x\}$  is an  $m_X$ -closed set or  $\{x\}$  is an  $n_X$ -open set.

*Proof.* (1)  $\Rightarrow$  (2). Let  $x \in X$ . If  $\{x\}$  is not  $m_X$ -closed, then  $A = X \setminus \{x\}$  is not  $m_X$ -open and hence,  $A$  is trivially  $mn$ - $\mathcal{J}g$ -closed. By (1),  $A$  is  $n_X$ -closed and hence,  $\{x\}$  is  $n_X$ -open.

(2)  $\Rightarrow$  (1). Let  $A \subset X$  be  $mn$ - $\mathcal{J}g$ -closed and let  $x \in A_n^*$ . We have the following two cases:

Case 1.  $\{x\}$  is  $m_X$ -closed. By Theorem 5.4,  $A_n^* \setminus A$  does not contain any nonempty  $m_X$ -closed. Since  $x \in A_n^*$ , then we obtain  $x \in A$ .

Case 2.  $\{x\}$  is  $n_X$ -open. We have  $X \setminus \{x\}$  is  $n_X$ -closed and hence:

$$(X \setminus \{x\})_n^* \subset X \setminus \{x\}$$

or equivalently

$$\{x\} \subset X \setminus (X \setminus \{x\})_n^*.$$

Since  $x \in \{x\}$ , then  $x \notin (X \setminus \{x\})_n^*$  and hence, there exists an  $n_X$ -open set  $U$  such that  $x \in U$  and  $U \cap (X \setminus \{x\}) \in \mathcal{J}$ . On the other hand,  $x \in A_n^*$  entails that  $V \cap A \notin \mathcal{J}$  for each  $n_X$ -open set such that  $x \in V$ . In particular, there exists  $U \in n_X$  such that  $x \in U$ ,  $U \cap A \notin \mathcal{J}$  and  $U \cap (X \setminus \{x\}) \in \mathcal{J}$ . We claim that  $\{x\} \cap A \neq \emptyset$ . If  $\{x\} \cap A = \emptyset$ , then  $A \subset X \setminus \{x\}$  and hence  $U \cap A \subset U \cap (X \setminus \{x\}) \in \mathcal{J}$ . It follows that  $U \cap A \in \mathcal{J}$ . This is a contradiction. Therefore,  $\{x\} \cap A \neq \emptyset$  and so  $x \in A$ .

Thus in both cases  $x$  is in  $A$  and hence  $A_n^* \subset A$ . Thus,  $A$  is  $n_X$ -closed, which shows that  $X$  is  $mn$ - $T_{\mathcal{J}}$ . □

**Definition 5.7.** [17] A bi  $m$ -space  $(X, m_X, n_X)$ , where  $n_X$  has property  $\mathcal{B}$ , is said to be  $mn$ - $T_{1/2}$  if and only if every singleton  $\{x\}$  of  $X$  is  $n_X$ -open or  $m_X$ -closed.

**Corollary 5.3.** *Let  $(X, m_X, n_X, \mathcal{J})$  an ideal bi  $m$ -space. If  $X$  is  $mn$ - $T_{1/2}$ , then  $X$  is  $mn$ - $T_{\mathcal{J}}$ .*

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*Universidad de Oriente*

*Núcleo de Sucre*

*Departamento de Matemáticas*

*Avenida Universidad, Cerro Colorado, Cumaná, Estado Sucre, Venezuela*

*E-mail address: jesanabri@gmail.com, jsanabria@udo.edu.ve*

*Universidad de Oriente*

*Núcleo de Sucre*

*Departamento de Matemáticas*

*Avenida Universidad, Cerro Colorado, Cumaná, Estado Sucre, Venezuela*

*E-mail address: ennisrafael@gmail.com*

*Universidad de Oriente*

*Núcleo de Sucre*

*Departamento de Matemáticas*

*Avenida Universidad, Cerro Colorado, Cumaná, Estado Sucre, Venezuela*

*E-mail address: carpintero.carlos@gmail.com*

*Universidad de Oriente*

*Núcleo de Sucre*

*Departamento de Matemáticas*

*Avenida Universidad, Cerro Colorado, Cumaná, Estado Sucre, Venezuela*

*E-mail address: salasbrown@gmail.com*