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SOME ITERATIVE ALGORITHMS FOR SOLVING MIXED VARIATIONAL INEQUALITIES

ABDELLAH BNOUHACHEM, M. ASLAM NOOR, ABDELLAH MASSAQ AND SHENG ZHAOHAN

Abstract. In this paper, we propose two new methods for solving mixed quasi variational inequalities by using the resolvent operator technique. Under certain conditions, the global convergence of the both methods is proved. The skew-symmetry of the nonlinear bifunction plays a crucial part in the convergence analysis of these new iterative methods. The comparison of these methods with other methods for solving the mixed quasi variational inequalities is an open interesting problem.

1. INTRODUCTION

Variational inequality has become a rich of inspiration in pure and applied mathematics. In recent years, classical variational inequality problems have been extended and generalized to study a large variety of problems arising in structural analysis, economics, optimization, operations research and engineering sciences, see [1-30] and the references therein. The projection and contraction method and its invariant forms represent an important tool for finding the approximation solution of various types of variational inequalities and complementarity problems.

In recent years variational inequalities theory has seen a dramatic increase in its applications and numerical methods. As a result of these activities, variational inequalities have been extended in various directions using novel and innovative techniques. A useful and important generalization of variational inequalities is called the mixed quasi variational inequality involving the nonlinear bifunction. Such type of mixed quasi variational inequalities arise in the study of elasticity with non-local friction laws, fluid flow through porous media and structural analysis. For the finite element analysis, existence results and applications, see [1, 16, 17, 20, 29]. Due to the presence of the nonlinear bifunction, the projection method and its variant forms including the Wiener-Hopf equations technique can not be extended to suggest iterative methods for solving mixed quasi variational inequalities. To overcome these drawbacks, some iterative methods have been suggested for special cases of the mixed quasi variational inequalities. For example, if the bifunction is proper, convex and lower semicontinuous function with respect to the first argument, then one can show that the mixed quasi variational inequalities are equivalent to the fixed-point problems and the implicit resolvent equations using the resolvent operator technique. This equivalent formulation has been used to suggest and analyze some iterative methods, the convergence of these methods requires

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that the operator is both strongly monotone and Lipschitz continuous. Secondly, it is very difficult to evaluate the resolvent of the operator expect for very simple cases. This fact has motivated many authors to develop the auxiliary principle technique, Lions and Stampacchia [18], Glowinski et al. [12] used this technique to study the existence of solution for the general mixed variational inequalities. In recent years, Noor [21, 22, 23, 25, 27] has extended the auxiliary principle technique to investigate the existence of a solution of various classes of variational inequalities and to suggest and analysis several algorithms for general mixed quasi variational inequalities. It has been shown that three-step iterative methods [3, 9] are more efficient than two-step and one-step iterative methods.

Inspired and motivated by the research going in this direction, we suggest and consider two new resolvent iterative methods for solving the mixed quasi variational inequalities involving the nonlinear term, which is the main motivation of this paper. We prove the global convergence of these new methods under some mild and suitable conditions. Since the mixed quasi variational inequalities includes the mixed variational inequalities, variational inequalities and complementarity problems as special cases, results obtained in this paper continue to hold for these problems. It is expected that these results may inspire and motivate others to find novel and innovative applications in various branches of pure and applied sciences.

2. PRELIMINARIES

Let H be a real Hilbert finite-dimensional space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, let I be the identity mapping on H, and $T\colon H\to H$ be nonlinear operators . Let $\partial\varphi$ denotes the subdifferential of function φ, where $\varphi\colon H\to R\cup\{+\infty\}$ is a proper convex lower semi continuous function on H. It is well known that the subdifferential $\partial\varphi$ is a maximal monotone operator. We consider the problem of finding $u\in H$ such that

$$\langle Tu, v-u\rangle+\varphi(v,u)-\varphi(u,u)\geq 0,\quad \forall\ v\in H. \tag{2.1}$$

Problem (2.1) is called the *mixed quasi variational inequality*. It has been shown that a large class of obstacle, unilateral, contact, free, moving, and equilibrium problems arising in regional, physical, mathematical, engineering and applied sciences can be studied in the unified and general framework of the mixed quasi variational inequalities (2.1).

For $\varphi(v,u)=\varphi(v)$, $\forall\ u\in H$, problem (2.1) reduces to finding $u\in H$ such that

$$\langle Tu, v-u\rangle+\varphi(v)-\varphi(u)\geq 0,\quad \forall\ v\in H,$$

which is called the mixed variational inequality or variational inequality of the second kind, see [1, 5, 6, 7, 12, 16, 17, 20, 29].

If $\varphi(\cdot,\cdot)=\varphi(\cdot)$ is an indicator function of a closed convex set K in H, then problem (2.1) is equivalent to finding $u\in K$ such that

$$\langle Tu, v-u\rangle\geq 0,\quad \forall\ v\in K,$$

is called as the classical variational inequality problem, and was introduced by Stampacchia [30] in 1964. For applications, numerical techniques and physical formulation, see [1-30].

We also need the following well known results and concepts.

Definition 2.1. $(\forall)u,v\in H$, the operator $T\colon H\to H$ is said to be *pseudomonotone*, if

$$\langle Tu, v-u\rangle\geq 0\quad\text{implies}\quad\langle Tv, v-u\rangle\geq 0.$$

It is well known [10] that monotonicity implies pseudomonotonicity, but the converse is not true. This shows that pseudomonotonicity is a weaker condition than monotonicity.

Definition 2.2. The bifunction $\varphi(\cdot,\cdot)$ is said to be *skew-symmetric,* if,

$$\varphi(u,u)-\varphi(u,v)-\varphi(v,u)+\varphi(v,v)\geq 0,\quad (\forall)u,v\in H.$$

Clearly, if the bifunction $\varphi(\cdot,\cdot)$ is linear in both arguments, then,

$$\varphi(u,u)-\varphi(u,v)-\varphi(v,u)+\varphi(v,v)=\varphi(u-v,u-v)\geq 0,\quad (\forall)u,v\in H,$$

which shows that the bifunction $\varphi(\cdot,\cdot)$ is nonnegative.

Definition 2.3. [8] Let A be a maximal monotone operator, then the *resolvent operator* associated with A is defined as

$$J_A(u)=(I+\rho A)^{-1}(u),\quad \forall\ u\in H,$$

where $\rho>0$ is a constant and I is the identity operator.

Remark 2.1. It is well known that the subdifferential $\partial\varphi(\cdot,\cdot)$ of a convex, proper and lower-semicontinuous function $\varphi(\cdot,\cdot)\colon H\times H\to R\cup\{+\infty\}$ is a maximal monotone with respect to the first argument, we can define its resolvent by

$$J_{\varphi(u)}=(I+\rho\partial\varphi(\cdot,u))^{-1})\equiv(I+\rho\partial\varphi(u))^{-1},\tag{2.2}$$

where $\partial\varphi(u)\equiv\partial\varphi(\cdot,u)$, unless otherwise specified.

The resolvent operator $J_{\varphi(u)}$defined by (2.2) has the following characterization,

Lemma 2.1. [25] *For a given $u\in H$, $z\in H$ satisfies the inequality*

$$\langle u-z,v-u\rangle+\rho\varphi(v,u)-\varphi(u,u)\geq 0,\quad \forall\ v\in H,$$

if and only if

$$u=J_{\varphi(u)}[z],$$

where $J_{\varphi(u)}$ is resolvent operator defined by (2.2).

It follows from Lemma 2.1 that

$$\langle J_{\varphi(u)}[z]-z,v-J_{\varphi(u)}[z]\rangle+\rho\varphi(v,J_{\varphi(u)}[z])-\rho\varphi(J_{\varphi(u)}[z],J_{\varphi(u)}[z])\geq 0,\quad (\forall)u,v,z\in H.\tag{2.3}$$

The following result can be proved by using Lemma 2.1.

Lemma 2.2. *u^* is solution of problem* (2.1) *if and only if $u^*\in H$ satisfies the relation:*

$$u^*=J_{\varphi(u^*)}[u^*-\rho T(u^*)],$$

where $\rho>0$.

From Lemma 2.2, it is clear that z is solution of (2.1) if and only if z is a zero point of the function

$$r(z,\rho):=z-J_{\varphi(u)}[z-\rho T(z)].$$

Lemma 2.3. [28] *For all $u\in H$ and $\rho'\geq\rho>0$, it holds that*

$$\|r(u,\rho')\|\geq\|r(u,\rho)\|\tag{2.4}$$

and

$$\frac{\|r(u,\rho')\|}{\rho'}\leq\frac{\|r(u,\rho)\|}{\rho}.\tag{2.5}$$

Lemma 2.4. *For all $v, w \in H$, we have*

$$\|J_{\varphi(u)}(w) - J_{\varphi(u)}(v)\|^2 \leq \langle w - v, J_{\varphi(u)}(w) - J_{\varphi(u)}(v)\rangle. \tag{2.6}$$

Proof. By using (2.3), we get

$$\langle w - J_{\varphi(u)}(w), J_{\varphi(u)}(w) - J_{\varphi(u)}(v)\rangle + \rho\varphi(J_{\varphi(u)}(v), J_{\varphi(u)}(w)) - \rho\varphi(J_{\varphi(u)}(w), J_{\varphi(u)}(w)) \geq 0 \tag{2.7}$$

and

$$\langle v - J_{\varphi(u)}(v), J_{\varphi(u)}(v) - J_{\varphi(u)}(w)\rangle + \rho\varphi(J_{\varphi(u)}(w), J_{\varphi(u)}(v)) - \rho\varphi(J_{\varphi(u)}(v), J_{\varphi(u)}(v)) \geq 0. \tag{2.8}$$

Adding (2.7) and (2.8), and using the skew-symmetry of the bifunction $\varphi(\cdot,\cdot)$, we obtain

$$\langle v - w, J_{\varphi(u)}(v) - J_{\varphi(u)}(w)\rangle \geq \|J_{\varphi(u)}(v) - J_{\varphi(u)}(w)\|^2,$$

and the proof is complete. □

Throughout this paper, we make following assumptions.

- T is continuous pseudomonotone operator on H.
- The bifunction $\varphi(\cdot,\cdot)$ is skew-symmetric.
- The solution set of problem (2.1) denoted by S^* is nonempty.

3. MAIN RESULTS

In this section, we suggest and analyze the new resolvent methods for solving mixed quasi variational inequalities (2.1). For given $u^k \in H$ and $\rho_k > 0$, each iteration of the first method consists of three steps, the first step offers $\tilde{u}^k$, the second step makes $\bar{u}^k$ and the third step produces the new iterate u^{k+1}.

Algorithm 3.1

Step 1. Given $u^0 \in H$, $\epsilon > 0$, $\rho_0 = 1$, $\nu > 1$, $\mu \in (0, \sqrt{2})$, $\tau \in (0,1)$, $\eta_1 \in (0,\tau)$, $\eta_2 \in (\tau,\nu)$ and let $k = 0$.

Step 2. If $\|r(u^k, 1)\| \leq \epsilon$, then stop. Otherwise, go to Step 3.

Step 3. 1) For a given $u^k \in H$, calculate the two predictors

$$\tilde{u}^k = J_{\varphi(u^k)}[u^k - \rho_k T(u^k)], \tag{3.1a}$$

$$\bar{u}^k = J_{\varphi(u^k)}[\tilde{u}^k - \rho_k T(\tilde{u}^k)]. \tag{3.1b}$$

2) If $\|r(\tilde{u}^k, 1)\| \leq \epsilon$, then stop . Otherwise, continue.

3) If ρ_k satisfies both

$$r_1 := \frac{\|\rho_k[\langle \tilde{u}^k - \bar{u}^k, T(u^k) - T(\tilde{u}^k)\rangle - \langle u^k - \bar{u}^k, T(\tilde{u}^k) - T(\bar{u}^k)\rangle]\|}{\|\tilde{u}^k - \bar{u}^k\|^2} \leq \mu^2 \tag{3.2}$$

and

$$r_2 := \frac{\|\rho_k(T(\tilde{u}^k) - T(\bar{u}^k))\|}{\|\tilde{u}^k - \bar{u}^k\|} \leq \nu, \tag{3.3}$$

then go to Step 4; otherwise, continue.

4) Perform an Armijo-like line search via reducing ρ_k

$$\rho_k := \rho_k * \frac{0.8}{\max(r_1, 1)}$$

and go to Step 3.

Step 4. Take the new iteration u^{k+1}, by setting

$$u^{k+1} = u^k - \alpha_k d(\tilde{u}^k, \bar{u}^k)), \tag{3.4}$$

where

$$\alpha_k = \frac{\langle u^k - \bar{u}^k, d(\tilde{u}^k, \bar{u}^k)\rangle}{\|d(\tilde{u}^k, \bar{u}^k)\|^2}$$

and

$$d(\tilde{u}^k, \bar{u}^k) := (\tilde{u}^k - \bar{u}^k) - \rho_k(T(\tilde{u}^k) - T(\bar{u}^k)). \tag{3.5}$$

Step 5. Adaptive rule of choosing a suitable ρ_{k+1} as the start prediction step size for the next iteration

1) Prepare a proper ρ_{k+1},

$$\rho_{k+1} := \begin{cases} \rho_k * \tau / r_2 & \text{if } r_2 \le \eta_1 \\ \rho_k * \tau / r_2 & \text{if } r_2 \ge \eta_2 \\ \rho_k & \text{otherwise.} \end{cases}$$

2) Return to Step 2, with k replaced by $k+1$.

We show that Algorithm 3.1 is well-defined. To see this, we need to show that the Armijo-like line search procedure is well defined.

Lemma 3.1. *In the kth iteration, if $\|r(u^k, 1)\| \ge \epsilon$, then the Armijo-like line search procedure with criteria* (3.2) *and* (3.3) *is finite.*

Proof. Assume for contradiction that ρ_k does not satisfy criterion (3.2) or (3.3) in finite Armijo-like line search procedure. Consequently, $\rho_k \to 0$, (see (3.4)). Without losing generality, we can assume $\rho_k < 1$.

Let us consider two possible cases.

Case 1. Criterion (3.2) fails to be satisfied. It follows that

$$\mu^2\|\tilde{u}^k - \bar{u}^k\|^2 < \|\rho_k[\langle \tilde{u}^k - \bar{u}^k, T(u^k) - T(\tilde{u}^k)\rangle - \langle u^k - \bar{u}^k, T(\tilde{u}^k) - T(\bar{u}^k)\rangle]\|.$$

This implies that either

$$\frac{1}{2}\mu^2\|\tilde{u}^k - \bar{u}^k\|^2 < \|\rho_k\langle \tilde{u}^k - \bar{u}^k, T(u^k) - T(\tilde{u}^k)\rangle\| \tag{3.6}$$

or

$$\frac{1}{2}\mu^2\|\tilde{u}^k - \bar{u}^k\|^2 < \|\rho_k\langle u^k - \bar{u}^k, T(\tilde{u}^k) - T(\bar{u}^k)\rangle\| \tag{3.7}$$

holds.

If (3.6) holds, by using the Cauchy-Schwarz inequality and dividing both sides of (3.6) by ρ_k, we have that

$$\frac{\mu^2\|\tilde{u}^k - \bar{u}^k\|}{2\rho_k} < \|T(u^k) - T(\tilde{u}^k)\|. \tag{3.8}$$

Note that

$$\|\tilde{u}^k - \bar{u}^k\| = \|\tilde{u}^k - J_{\varphi(u^k)}[\tilde{u}^k - \rho_k T(\tilde{u}^k)]\| = \|r(\tilde{u}^k, \rho_k)\|,$$

substituting above equality into (3.8) and using inequality (2.5), we find that

$$\frac{1}{2}\mu^2\|r(\tilde{u}^k,1)\| \leq \frac{\mu^2\|r(\tilde{u}^k,\rho_k)\|}{2\rho_k} < \|T(u^k)-T(\tilde{u}^k)\|. \tag{3.9}$$

It is easy to see that $\tilde{u}^k \to u^k$ (since $\rho_k \to 0$). Consequently, $T(\tilde{u}^k) \to T(u^k)$ and $r(\tilde{u}^k,1) \to r(u^k,1)$ due to continuity of $T(u)$ and $r(u,1)$, respectively. When we take $\rho_k \to 0$ in (3.9), we get $\|r(u^k,1)\| \leq 0$. But this contradicts the assertion that $\epsilon \leq \|r(u^k,1)\|$.

Let us turn to deal with (3.7). Note that

$$\|u^k-\tilde{u}^k\| = \|u^k - J_{\varphi(u^k)}[u^k-\rho_k T(u^k)]\| = \|r(u^k,\rho_k)\|,$$

then we have

$$\|u^k-\bar{u}^k\| \leq \|u^k-\tilde{u}^k\| + \|\tilde{u}^k-\bar{u}^k\| = \|r(u^k,\rho_k)\| + \|r(\tilde{u}^k,\rho_k)\|. \tag{3.10}$$

In (3.7), using Cauchy-Schwarz inequality and (3.10), we get immediately,

$$\frac{1}{2}\mu^2\|\tilde{u}^k-\bar{u}^k\|^2 < \rho_k\|u^k-\bar{u}^k\|\|T(\tilde{u}^k)-T(\bar{u}^k)\| \leq \rho_k(\|r(u^k,\rho_k)\|+\|r(\tilde{u}^k,\rho_k)\|)\|T(\tilde{u}^k)-T(\bar{u}^k)\|. \tag{3.11}$$

Dividing both sides of (3.11) by ρ_k^2, using the equality (3.9) and inequality (2.5) again, we obtain

$$\frac{1}{2}\mu^2\|r(\tilde{u}^k,1)\|^2 \leq \frac{\mu^2\|r(\tilde{u}^k,\rho_k)\|^2}{2\rho_k^2} < \left(\frac{\|r(u^k,\rho_k)\|}{\rho_k} + \frac{\|r(\tilde{u}^k,\rho_k)\|}{\rho_k}\right)\|T(\tilde{u}^k)-T(\bar{u}^k)\|.$$

It is easy to see that $\tilde{u}^k \to u^k$, $\bar{u}^k \to u^k$ (since $\rho_k \to 0$). Consequently, we can find $\delta_1 > 0$, $\delta_2 > 0$ such that $\|r(u^k,\rho_k)\| < \rho_k\delta_1$ and $\|r(\tilde{u}^k,\rho_k)\| < \rho_k\delta_2$. We have also $T(\tilde{u}^k) \to T(u^k)$, $T(\bar{u}^k) \to T(u^k)$ and $r(\tilde{u}^k,1) \to r(u^k,1)$ due to continuity of $T(u)$ and $r(u,1)$, respectively. By taking $\rho_k \to 0$ in above inequality, we obtain $\|r(u^k,1)\| \leq 0$. Therefore, $\|r(u^k,1)\| = 0$, contradicting that u^k is not a solution.

Case 2. Condition (3.3) is violated. Then we must have

$$\nu\|\tilde{u}^k-\bar{u}^k\| < \|\rho_k(T(\tilde{u}^k)-T(\bar{u}^k))\|. \tag{3.12}$$

The proof is quite similar to the Case 1. Dividing both sides of (3.12) by ρ_k and taking $\rho_k \to 0$, we get the contradiction.

From the above observations, we assert that Algorithm 3.1 is well-defined. □

Lemma 3.2. *Let u^* be a solution of problem* (2.1). *For given $u^k \in H$, let $\tilde{u}^k$, $\bar{u}^k$ be the predictors produced by* (3.1a) *and* (3.1b), *then we have*

$$\langle u^k-\bar{u}^k, d(\tilde{u}^k,\bar{u}^k)\rangle \geq (2-\mu^2)\|\tilde{u}^k-\bar{u}^k\|^2. \tag{3.13}$$

Proof. Note that $\tilde{u}^k = J_{\varphi(u^k)}[u^k-\rho_k T(u^k)]$, $\bar{u}^k = J_{\varphi(u^k)}[\tilde{u}^k-\rho_k T(\tilde{u}^k)]$, we can apply (2.6) with $v = u^k-\rho_k T(u^k)$, $w = \tilde{u}^k-\rho_k T(\tilde{u}^k)$ to obtain

$$\langle u^k-\rho_k T(u^k)-(\tilde{u}^k-\rho_k T(\tilde{u}^k)), \tilde{u}^k-\bar{u}^k\rangle \geq \|\tilde{u}^k-\bar{u}^k\|^2.$$

By some manipulations, we have

$$\langle u^k-\tilde{u}^k, \tilde{u}^k-\bar{u}^k\rangle \geq \|\tilde{u}^k-\bar{u}^k\|^2 + \rho_k\langle \tilde{u}^k-\bar{u}^k, T(u^k)-T(\tilde{u}^k)\rangle.$$

Then, we obtain

$$\begin{aligned}\langle u^k-\tilde{u}^k, d(\tilde{u}^k,\bar{u}^k)\rangle &= \langle u^k-\tilde{u}^k, \tilde{u}^k-\bar{u}^k\rangle - \rho_k\langle u^k-\tilde{u}^k, T(\tilde{u}^k)-T(\bar{u}^k)\rangle \\ &\geq \|\tilde{u}^k-\bar{u}^k\|^2 + \rho_k\langle \tilde{u}^k-\bar{u}^k, T(u^k)-T(\tilde{u}^k)\rangle \\ &\quad -\rho_k\langle u^k-\tilde{u}^k, T(\tilde{u}^k)-T(\bar{u}^k)\rangle. \end{aligned} \tag{3.14}$$

Using (3.14), (3.2) and the definition of $d(\tilde{u}^k,\bar{u}^k)$, we get

$$\begin{aligned}\langle u^k-\bar{u}^k, d(\tilde{u}^k,\bar{u}^k)\rangle &= \langle u^k-\tilde{u}^k, d(\tilde{u}^k,\bar{u}^k)\rangle + \langle \tilde{u}^k-\bar{u}^k, d(\tilde{u}^k,\bar{u}^k)\rangle \\ &\geq \|\tilde{u}^k-\bar{u}^k\|^2 + \rho_k\langle \tilde{u}^k-\bar{u}^k, T(u^k)-T(\tilde{u}^k)\rangle \\ &\quad -\rho_k\langle u^k-\tilde{u}^k, T(\tilde{u}^k)-T(\bar{u}^k)\rangle + \|\tilde{u}^k-\bar{u}^k\|^2 \\ &\quad -\rho_k\langle \tilde{u}^k-\bar{u}^k, T(\tilde{u}^k)-T(\bar{u}^k)\rangle \\ &\geq (2-\mu^2)\|\tilde{u}^k-\bar{u}^k\|^2. \end{aligned}$$

Hence, (3.13) holds and the proof is completed. □

Now, we mainly focus on investigating the convergence of Algorithm 3.1. The following theorem plays a crucial role in the convergence of Algorithm 3.1.

Theorem 3.1. *Let u^* be a solution of problem* (2.1) *and let u^{k+1} be the sequence obtained from algorithm* 3.1. *Then u^k is bounded and*

$$\|u^{k+1}-u^*\|^2 \leq \|u^k-u^*\|^2 - \frac{(2-\mu^2)^2}{(1+\nu)^2}\|\tilde{u}^k-\bar{u}^k\|^2. \tag{3.15}$$

Proof. For any $u^* \in S^*$ solution of problem (2.1), we have

$$\langle \rho_k T(u^*), \bar{u}^k-u^*\rangle + \rho_k\varphi(\bar{u}^k,u^*) - \rho_k\varphi(u^*,u^*) \geq 0.$$

Using the pseudomonotonicity of T, we obtain

$$\langle \rho_k T(\bar{u}^k), \bar{u}^k-u^*\rangle + \rho_k\varphi(\bar{u}^k,u^*) - \rho_k\varphi(u^*,u^*) \geq 0. \tag{3.16}$$

Substituting $z=\tilde{u}^k-\rho_k T(\tilde{u}^k)$ and $v=u^*$ into (2.3), we get

$$\langle \tilde{u}^k-\rho_k T(\tilde{u}^k)-\bar{u}^k, \bar{u}^k-u^*\rangle + \rho_k\varphi(u^*,\bar{u}^k) - \rho_k\varphi(\bar{u}^k,\bar{u}^k) \geq 0. \tag{3.17}$$

Adding (3.16) and (3.17), and using the skew-symmetry of the bifunction $\varphi(\cdot,\cdot)$ and the definition of $d(\tilde{u}^k,\bar{u}^k)$, we have

$$\langle d(\tilde{u}^k,\bar{u}^k), \bar{u}^k-u^*\rangle \geq 0. \tag{3.18}$$

Since $u^* \in H$ be a solution of problem (2.1), then

$$\begin{aligned}\|u^{k+1}-u^*\|^2 &= \|u^k-u^*-\alpha_k d(\tilde{u}^k,\bar{u}^k)\|^2 \\ &= \|u^k-u^*\|^2 - 2\alpha_k\langle u^k-u^*, d(\tilde{u}^k,\bar{u}^k)\rangle + \alpha_k^2\|d(\tilde{u}^k,\bar{u}^k)\|^2. \end{aligned} \tag{3.19}$$

Adding (3.18) (multiplied by $2\alpha_k$) to (3.19) and using (3.5)

$$\begin{aligned}\|u^{k+1}-u^*\|^2 &= \|u^k-u^*\|^2 - 2\alpha_k\langle u^k-\bar{u}^k, d(\tilde{u}^k,\bar{u}^k)\rangle + \alpha_k^2\|d(\tilde{u}^k,\bar{u}^k)\|^2 \\ &\leq \|u^k-u^*\|^2 - \alpha_k\langle u^k-\bar{u}^k, d(\tilde{u}^k,\bar{u}^k)\rangle \\ &\leq \|u^k-u^*\|^2 - \alpha_k(2-\mu^2)\|\tilde{u}^k-\bar{u}^k\|^2, \end{aligned} \tag{3.20}$$

where the last inequality follows from (3.13)

Recalling the definition of $d(\tilde{u}^k, \bar{u}^k)$ (see (3.5)) and applying Criterion (3.3), it is easy to see that

$$\|d(\tilde{u}^k, \bar{u}^k)\|^2 \leq (\|\tilde{u}^k - \bar{u}^k\| + \|\rho_k(T(\tilde{u}^k) - T(\bar{u}^k))\|)^2 \leq (1+\nu)^2\|\tilde{u}^k - \bar{u}^k\|^2. \tag{3.21}$$

Moreover, by using (3.13) together with (3.21), we get

$$\alpha_k = \frac{\langle u^k - \bar{u}^k, d(\tilde{u}^k, \bar{u}^k)\rangle}{\|d(\tilde{u}^k, \bar{u}^k)\|^2} \geq \frac{2-\mu^2}{(1+\nu)^2} > 0, \quad \mu \in (0, \sqrt{2}). \tag{3.22}$$

Substituting (3.22) in (3.20), we get the assertion of this theorem. Since $\gamma \in [1, 2)$ and $\mu \in (0, \sqrt{2})$ we have

$$\|u^{k+1} - u^*\| \leq \|u^k - u^*\| \leq \cdots \leq \|u^0 - u^*\|.$$

Then the sequence u^k is bounded. □

We now present the convergence result of Algorithm 3.1.

Theorem 3.2. *If* $\inf_{k=0}^{\infty} \rho_k := \rho > 0$, *then any cluster point of the sequence* $\{\tilde{u}^k\}$ *generated by Algorithm* 3.1 *is a solution of problem* (2.1).

Proof. It follows from (3.15) that

$$\lim_{k\to\infty} \|\tilde{u}^k - \bar{u}^k\| = 0.$$

Since the sequence u^k is bounded, $\{\tilde{u}^k\}$ is also bounded, it has at least a cluster point. Let u^∞ be a cluster point of $\{\tilde{u}^k\}$ and the subsequence $\{\tilde{u}^{k_j}\}$ converges to u^∞. Using the continuity of $r(u, \rho)$ and inequality (2.4), it follows that

$$\|r(u^\infty, \rho)\| = \lim_{k_j\to\infty} \|r(\tilde{u}^{k_j}, \rho)\| \leq \lim_{k_j\to\infty} \|r(\tilde{u}^{k_j}, \rho_{k_j})\| = \lim_{k_j\to\infty} \|\tilde{u}^{k_j} - \bar{u}^{k_j}\| = 0.$$

This means that u^∞ is a solution of problem (2.1). □

Let $w^k = u^k - \alpha_k\rho_k T(\tilde{u}^k)$. For a positive constant τ, we consider

$$u^{k+1} = u^k - \tau(u^k - \bar{u}^k).$$

Here the positive constant τ can be viewed as a step along the direction $-(u^k - \bar{u}^k)$.

We suggest the following iterative method.

Algorithm 3.2

Step 1. Given $u^0 \in H$, $\epsilon > 0$, $\rho_0 = 1$, $\nu > 1$, $\mu \in (0, \sqrt{2})$, $\tau \in (0, 1)$, $\eta_1 \in (0, \tau)$, $\eta_2 \in (\tau, \nu)$ and let $k = 0$.

Step 2. If $\|r(u^k, 1)\| \leq \epsilon$, then stop. Otherwise, go to Step 3.

Step 3. 1) For a given $u^k \in H$, calculate the two predictors

$$\tilde{u}^k = J_{\varphi(u^k)}[u^k - \rho_k T(u^k)],$$

$$\bar{u}^k = J_{\varphi(u^k)}[\tilde{u}^k - \rho_k T(\tilde{u}^k)].$$

2) If $\|r(\tilde{u}^k, 1)\| \leq \epsilon$, then stop . Otherwise, continue.

3) If ρ_k satisfies both

$$r_1 := \frac{\|\rho_k[\langle \tilde{u}^k - \bar{u}^k, T(u^k) - T(\tilde{u}^k)\rangle - \langle u^k - \bar{u}^k, T(\tilde{u}^k) - T(\bar{u}^k)\rangle]\|}{\|\tilde{u}^k - \bar{u}^k\|^2} \leq \mu^2$$

and

$$r_2 := \frac{\|\rho_k(T(\tilde{u}^k) - T(\bar{u}^k))\|}{\|\tilde{u}^k - \bar{u}^k\|} \leq \nu,$$

then go to Step 4; otherwise, continue.

4) Perform an Armijo-like line search via reducing ρ_k

$$\rho_k := \rho_k * \frac{0.8}{\max(r_1, 1)}$$

and go to Step 3.

Step 4. Compute

$$w^k = u^k - \alpha_k d(\tilde{u}^k, \bar{u}^k)),$$

where

$$\alpha_k = \frac{\langle u^k - \bar{u}^k, d(\tilde{u}^k, \bar{u}^k)\rangle}{\|d(\tilde{u}^k, \bar{u}^k)\|^2}$$

and

$$d(\tilde{u}^k, \bar{u}^k) := (\tilde{u}^k - \bar{u}^k) - \rho_k(T(\tilde{u}^k) - T(\bar{u}^k)).$$

Step 5. For $\tau > 0$, the new iterate $u^{k+1}(\tau)$ is defined by

$$u^{k+1}(\tau) = u^k - \tau(u^k - w^k). \tag{3.23}$$

Step 6. Adaptive rule of choosing a suitable ρ_{k+1} as the start prediction step size for the next iteration

1) Prepare a proper ρ_{k+1},

$$\rho_{k+1} := \begin{cases} \rho_k * \tau/r_2 & \text{if } r_2 \leq \eta_1 \\ \rho_k * \tau/r_2 & \text{if } r_2 \geq \eta_2 \\ \rho_k & \text{otherwise.} \end{cases}$$

2) Return to Step 2, with k replaced by $k + 1$.

How to choose a suitable step length $\tau > 0$ to force convergence will be discussed later. We now consider the criteria of τ, which ensures that $u^{k+1}(\tau)$ is closer to u^* than u^k. For this purpose, we define

$$\Gamma(\tau) := \|u^k - u^*\|^2 - \|u^{k+1}(\tau) - u^*\|^2.$$

Lemma 3.3. *Let $u^* \in S^*$ and $w^k = u^k - \alpha_k d(\tilde{u}^k, \bar{u}^k))$. Then we have*

$$\Gamma(\tau) = \tau\{\|u^k - w^k\|^2 + \Upsilon(\alpha_k)\} - \tau^2\|u^k - w^k\|^2, \tag{3.24}$$

where

$$\Upsilon(\alpha_k) := \|u^k - u^*\|^2 - \|w^k - u^*\|^2.$$

Proof. It follows from (3.23) that

$$\begin{aligned} \Gamma(\tau) &= \|u^k - u^*\|^2 - \|u^k - \tau(u^k - w^k) - u^*\|^2 \\ &= 2\tau\langle u^k - u^*, u^k - w^k\rangle - \tau^2\|u^k - w^k\|^2 \\ &= 2\tau\{\|u^k - w^k\|^2 - \langle u^* - w^k, u^k - w^k\rangle\} - \tau^2\|u^k - w^k\|^2. \end{aligned}$$

Using the following identity

$$\langle u^* - w^k, u^k - w^k\rangle = \frac{1}{2}\left(\|w^k - u^*\|^2 - \|u^k - u^*\|^2\right) + \frac{1}{2}\|u^k - w^k\|^2,$$

and the notation of $\Upsilon(\alpha_k)$, we obtain (3.24), the required result. □

Using (3.20) (by setting $w^k = u^{k+1}$) and (3.24), we get

$$\Gamma(\tau) \geq \Lambda(\tau), \tag{3.25}$$

where

$$\Lambda(\tau) = \tau\{\|u^k - w^k\|^2 + \alpha_k\langle u^k - \bar{u}^k, d(\tilde{u}^k, \bar{u}^k)\rangle\} - \tau^2\|u^k - w^k\|^2.$$

The above inequality tells us how to choose a suitable τ_k. Since $\Lambda(\tau_k)$ is a quadratic function of τ_k and it reaches its maximum at

$$\tau_k^* = \frac{\|u^k - w^k\|^2 + \alpha_k\langle u^k - \bar{u}^k, d(\tilde{u}^k, \bar{u}^k)\rangle}{2\|u^k - w^k\|^2}$$

and

$$\Lambda(\tau_k^*) = \frac{\tau_k^*\{\|u^k - w^k\|^2 + \alpha_k\langle u^k - \bar{u}^k, d(\tilde{u}^k, \bar{u}^k)\rangle\}}{2}.$$

Then, from Lemma 3.2 and (3.22), we get

$$\tau_k^* \geq \frac{1-\delta}{2}\left(\frac{\|u^k - w^k\|^2 + \frac{(2-\mu^2)^2}{(1+\nu)^2}\|\tilde{u}^k - \bar{u}^k\|^2}{2\|u^k - w^k\|^2}\right) \geq \frac{1-\delta}{4},$$

and

$$\begin{aligned}\Lambda(\tau_k^*) &\geq \frac{\tau_k^*(2-\mu^2)^2}{2(1+\nu)^2}\|\tilde{u}^k - \bar{u}^k\|^2 \\ &\geq \frac{(1-\delta)(2-\mu^2)^2}{8(1+\nu)^2}\|\tilde{u}^k - \bar{u}^k\|^2. \end{aligned}\tag{3.26}$$

Then, from (3.24), (3.25) and (3.26), we have

$$\begin{aligned}\|u^{k+1} - u^*\|^2 &= \|u^k - u^*\|^2 - \Gamma(\tau_k^*) \\ &\leq \|u^k - u^*\|^2 - \frac{(1-\delta)(2-\mu^2)^2}{8(1+\nu)^2}\|\tilde{u}^k - \bar{u}^k\|^2.\end{aligned}$$

The convergence of Algorithm 3.2 can be proved by similar arguments as Algorithm 3.1. Hence the proof is omitted.

Remark 3.1. If $\tau_k^* = 1$ Algorithm 3.2 reduces to Algorithm 3.1. Since τ_k^* is to maximize the profit function $\Lambda(\tau)$, we have

$$\Lambda(\tau_k^*) \geq \Lambda(1). \tag{3.27}$$

Inequalities (3.25) and (3.27) show theoretically that Algorithm 3.2 is expected to make more progress than Algorithm 3.1 at each iteration, and so it explains theoretically that Algorithm 3.2 outperforms Algorithm 3.1.

4. CONCLUSIONS

In this paper, we suggest and analyze two new methods for solving mixed quasi variational inequalities, which can be viewed as a refinement and improvement of some existing resolvent methods and projection descent methods. It is easy to verify that Algorithm 3.1 and Algorithm 3.2 include some existing methods (e.g. [25, 27, 28]) as special cases. Therefore, the new algorithms are expected to be widely applicable.

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REFERENCES

[1] C. Baiocchi and A. Capelo: *Variational and Quasi Variational Inequalities*, Wiley, New York, 1984.
[2] A. Bnouhachem: *A self-adaptive method for solving general mixed variational inequalities*, J. Math. Anal. Appl., **309**(2005), 136-150.
[3] A. Bnouhachem, M. A. Noor and Th. R. Rassias: *Three-steps iterative algorithms for mixed variational inequalities*, Appl. Math. Comput., **183**(2006), 436-446.
[4] A. Bnouhachem and M. A. Noor: *Numerical methods for general mixed variational inequalities*, Appl. Math. Comput., **204**(2008), 27-36.
[5] A. Bnouhachem, M. Li, M. Khalfaoui and S. Zhaohan: *A modified inexact implicit method for mixed variational inequalities*, J. Comput. Appl. Math., **234**(2010), 3356-3365.
[6] A. Bnouhachem, M. A. Noor and S. Zhaohan: *New three-step iterative method for solving mixed variational inequalities*, Int. J. Comput. Meth., **8**(2011), No. 1, 139-150.
[7] A. Bnouhachem, M. A. Noor and S. Zhaohan: *Some new resolvent methods for solving general mixed variational inequalities*, Int. J. Mod. Phys. B., **25**(2011), No. 32, 4419-4434.
[8] H. Brezis: *Operateur Maximaux Monotones et Semigroupes de Contractions dans les Espaces de Hilbert*, North-Holland, Amsterdam, Holland, 1973.
[9] X. Fu: *A two-stage prediction-correction method for solving variational inequlities*, J. Comp. Appl. Math., **214**(2008), 345-355.
[10] F. Giannessi and A. Maugeri: *Variational Inequalities and Network Equilibrium Problems*, Plenum Press, New York, 1995.
[11] F. Giannessi, A. Maugeri and P. M. Pardalos: *Equilibrium Problems. Nonsmooth Optimization and Variational Inequality Models*, Kluwer Academic Press, Dordrecht, Holland, 2001.
[12] R. Glowinski, J. L. Lions and R. Tremolieres: *Numerical Analysis of Variational Inequalities*, North-Holland, Amsterdam, Holland, 1981.
[13] P. T. Harker and J. S. Pang: *Finite-dimensional variational inequality and nonlinear complementarity problems (A survey of theory, algorithms and applications)*, Math. Program., **48**(1990), 161-220.
[14] B. S. He: *A class of projection and contraction methods for monotone variational inequalities*, Appl. Math. Optim., **35**(1997), 69-76.
[15] B. S. He, L. Z. Liao: *Improvement of some projection methods for monotone variational inequalities*, J. Optim. Theory Appl., **112**(2002), 111-128.
[16] N. Kikuchi and J. T. Oden: *Contact Problems in Elasticity*, SIAM, Philadelphia, 1988.
[17] D. Kinderlehrer and G. Stampacchia: *An Introduction to Variational Inequalities and Their Applications*, SIAM, Philadelphia, 2000.
[18] J. L. Lions and G. Stampacchia: *Variational inequalities*, Comm. Pure Apl. Math., **20**(1967), 493-512.
[19] M. A. Noor: *A new iterative method for monotone mixed variational inequalities*, Math. Comput. Modelling, **26**(1997), No. 7, 29-34.
[20] M. A. Noor: *Set-valued mixed quasi variational inequalities and implicit resolvent equations*, Math. Comput. Modelling, **29**(1999), 1-11.
[21] M. A. Noor: *A class of new iterative methods for general mixed variational inequalities*, Math. Comput. Modelling, **31**(2000), 11-19.

[22] M. A. Noor: *Some predictor-corrector algorithms for multivalued variational inequalities*, J. Optim. Theory Appl., **108**(2001), 659-670.
[23] M. A. Noor: *Proximal methods for mixed quasi variational inequalities*, J. Optim. Theory Appl., **115**(2002), 447-451.
[24] M. A. Noor: *Pseudomonotone general mixed variational inequalities*, Appl. Math. Comput., **141**(2003), 529-540.
[25] M. A. Noor: *Mixed quasi variational inequalities*, Appl. Math. Comput., **146**(2003), 553-578.
[26] M. A. Noor: *New extragradient-type methods for general variational inequalities*, J. Math. Anal. Appl., **277**(2003), 379-395.
[27] M. A. Noor: *Fundamentals of mixed quasi variational inequalities*, Int. J. Pure Appl. Math., **15**(2004), 137-258.
[28] M. A. Noor and A. Bnouhachem: *Self-adaptive methods for mixed quasi variational inequalities*, J. Math. Anal. Appl., **312**(2005), 514-526.
[29] J. T. Oden and N. Kikuchi: *Theory of variational inequalities with applications to problems of flow through porous media*, Int. J. Engng. Sci., **18**(1980), 1173-1284.
[30] G. Stampacchia: *Formes bilineaires coercivites sur les ensembles convexes*, C. R. Acad. Sci. Paris, **258**(1964), 4413-4416.

Nanjing University
School of Management Science and Engineering
210093 *Nanjing, P. R. China*
Ibn Zohr University
National School of Applied Sciences
BP 1136*, Agadir, Morocco*
E-mail address: babedallah@yahoo.com

COMSATS Institute of Information Technology
Mathematics Department
Islamabad, Pakistan
E-mail address: noormaslam@hotmail.com

Ibn Zohr University, Faculty of Sciences
Informatics Department
BP 8106*, Agadir, Morocco*

Nanjing University,
School of Management Science and Engineering,
210093 *Nanjing, P. R. China*