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A FIXED POINT THEOREM OF INTEGRAL TYPE

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Abstract. The theorem proved in this paper is an integral analogue of a previous Banach type fixed point theorem proved by Hicks and the author. The result of this paper provides a correct proof of a similar result appearing in a paper of Gariola and Rawat.

Let (X, d) be a complete metric space, T a selfmap of $X, x \in X$. The orbit of x, written $O(x)$, is defined by $O(x) := \{x, Tx, T^2x, \ldots\}$. A map $G\colon X \to \mathbb{R}$ is called T-orbitally lower semi-continuous at a point $q \in X$ if, for any sequence $\{x_n\} \subset O(x)$ with $x_n \to q$, one has $G(q) \leq \liminf G(x_n)$.

In 1979 Hicks and the author [4] proved the following result.

Theorem 1.1. *Let $0 \leq h < 1$. Suppose there exist an x such that*

(A) *$d(Ty, T^2y) \leq hd(y, Ty)$ for every $y \in O(x)$. Then*

(i) *$\lim T^n x = q$ exists;*

(ii) *$d(T^n x, q) \leq \dfrac{h^n}{1-h} d(x, Tx)$;*

(iii) *q is a fixed point of T is and only if $G(x) = d(x, Tx)$ is T-orbitally lower semi-continuous at q.*

In order to prove the integral version of the above Theorem, it is necessary to have the integral inequality, corresponding to (A), be true for every pair of points x and y in the orbit of u, instead of just adjacent points in the orbit, because the triangular inequality is not available.

Theorem 1.2. *Let T be a selfmap of a complete metric space, h a constant satisfying $0 \leq h < 1$, and suppose that there exists a point $u \in X$ such that*

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq h \int_0^{d(x,y)} \varphi(t)dt \tag{1.1}$$

for each $x, y \in O(u)$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is Lebesgue integrable, summable, and such that

$$\int_0^{\epsilon} \varphi(t)dt > 0 \quad for\, each \quad \epsilon > 0. \tag{1.2}$$

Then

(i) *$\lim\limits_n T^n x = q$ exists;*

(ii) *q is a fixed point of T if and only if $G(x) := d(x, Tx)$ is T-orbitally lower semi-continuous.*

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Proof. Let $x \in O(u)$. From (i),

$$\int_0^{d(T^n x, T^{n+1}x)} \varphi(t)dt \le h \int_0^{d(T^{n-1}x, T^n x)} \varphi(t)dt$$
$$\le \cdots \le h^n \int_0^{d(x,Tx)} \varphi(t)dt. \tag{1.3}$$

Then,

$$\int_0^{\lim_n d(T^n x, T^{n+1}x)} \varphi(t)dt = \lim_n \int_0^{d(T^x, T^{n+1}x)} \varphi(t)dt = 0,$$

which, using (1.2), implies that $\lim_n d(T^n x, T^{n+1}x) = 0$.

To prove that $\{T^n x\}$ is Cauchy, assume that it is not Then there exists a $\delta > 0$ and two subsequences $\{n(k)\}$ and $\{m(k)\}$, with $m(k) < n(k) < m(k+1)$ such that

$$d(T^{m(k)}x, T^{n(k)}x) \ge \delta \quad \text{and} \quad d(T^{m(k)}x, T^{n(k)-1}x) < \delta. \tag{1.4}$$

From (1.4), with $d_n := d(T^n x, T^{n+1}x)$,

$$\delta \le d(T^{m(x)}x, T^n x) \le d_{n(k)-1} + d(T^{m(k)-1}x, T^{n(k)-1}x) + d_{n(k)-1}$$
$$\le 2d_{n(k)-1} + d(T^{m(k)}x, T^{n(x)-1}x) + d_{n(k)-1}.$$

Taking the limit of the above as $k \to \infty$, it follows that

$$\lim_k d(T^{m(k)-1}x, T^{n(k)-1}x) = \delta.$$

From (1.1),

$$\int_0^{d(T^{m(k)}x, T^{n(k)}x)} \varphi(t)dt \le h \int_0^{d(T^{m(k)-1}x, T^{n(k)-1}x)} \varphi(t)dt.$$

Taking the limit of both sides of the inequality as $k \to \infty$ yields

$$\int_0^{\delta} \varphi(t)dt \le h \int_0^{\delta} \varphi(t)dt < \int_0^{\delta} \varphi(t)dt,$$

which, from (1.2), is a contradiction. Therefore $\{T^n x\}$ is Cauchy. Since X is complete, the sequence converges to some point q.

To prove (ii), if q is a fixed point of T and $\{x_n\}$ is any sequence in the orbit of u with $x_n \to q$, then

$$0 = G(q) \le \liminf G(x_n).$$

Conversely, if G is T-lower semi-continuous at q, then, using (1.3),

$$\int_0^{d(q,Tq)} \varphi(t)dt \le \liminf \int_0^{d(T^n x, T^{n+1}x)} \varphi(t)dt$$
$$\le \liminf h^n \int_0^{d(x,Tx)} \varphi(t)dt = 0.$$

□

Remarks:

1. The following is the statement of Theorem 2.1 of Gariola and Rawat [3].

Theorem 1.3. *Let $0 \leq \alpha < 1$. Suppose that there exists an x such that*

$$\int_0^{d(Ty,T^2y)} \varphi(t)dt \leq \alpha \int_0^{d(y,Ty)} \varphi(t)dt$$

for every $y \in O(x)$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}$ is a Lebesgue integrable mapping which is summable, nonnegative, and such that

$$\int_0^{\epsilon} \varphi(t)dt > 0 \quad for\ each \quad \epsilon > 0.$$

Then

(i) $\lim T^n x = q$ *exists.*

(ii) $\int_0^{d(t^n x,q)} \varphi(t)dt \leq \dfrac{\alpha^n}{1-\alpha} \int_0^{d(x,Tx)} \varphi(t)dt.$

(iii) *q is a fixed point of T if and only if $G(x) = d(x,Tx)$ is T-orbitally lower semi-continuous at q.*

In the proof of (i) the authors assume the integral version of the triangular inequality; i.e., if $a, b > 0$, then

$$\int_0^{a+b} \varphi(t)dt \leq \int_0^{a} \varphi(t)dt + \int_0^{b} \varphi(t)dt. \tag{1.5}$$

Since (1.5) is not true for every nonnegative integrable function ϕ, the proof of (i) is not correct.

The proof of (ii) also uses incorrect inequality (1.4).

2. In [2], Dey et al, in attempting to prove a fixed point theorem involving a contractive condition of integral type, also use incorrect inequality (1.5).

Aydi [1], in correcting the proof of [2], restricts the theorem to the class of subadditive functions φ; i.e., those for which inequality (1.4) is true.

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