J. Adv. Math. Stud.
Vol. **5**(2012), No. 2, 101-106
http://journal.fairpartners.ro

UNIFORM POLYNOMIAL TRICHOTOMY OF EVOLUTION OPERATORS IN BANACH SPACES

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Abstract. This paper presents necessary and sufficient conditions for uniform polynomial trichotomy of evolution operators in Banach spaces. Well-known results for uniform exponential trichotomy are extended to the case of uniform polynomial trichotomy.

1. INTRODUCTION

In the mathematical literature of the last decades, the asymptotic behaviour of solutions of evolution equations in infinite dimensional spaces has proved to be a research area of large intensity (see [1] - [5], [10], [11].)

The concept of trichotomy was introduced in 1967 by R. J. Sacker and G. R. Sell and studied for the case of linear differential equations in the finite dimensional setting in [13]. The main idea in the study of trichotomy is to obtain, at any moment, a decomposition of the state space in three subspaces: a stable subspace, an instable one and a third one, called the central manifold.

In this paper we consider two concepts of trichotomy for evolution operators in Banach spaces. The case of exponential trichotomy was studied by M. Megan and C. Stoica in [7], [9], and M. Megan, C. Stoica, L. Buliga in [9]. The case of polynomial stability, respectively polynomial dichotomy was studied by M. Megan, T. Ceauşu and M. L. Rămneanţu in [6], respectively [12].

We focus on studying the uniform polynomial trichotomy of evolution operators in Banach spaces. Our approach is based on the extension of techniques for uniform exponential trichotomy in the case of uniform polynomial trichotomy. We give an integral characterization for uniform polynomial trichotomy and we establish a connection between uniform polynomial trichotomy and uniform exponential trichotomy.

2. PRELIMINARIES

Let X be real or complex Banach space and let I be the identity operator on X. The norm on X and on $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators acting on X, will be denoted by $\|\cdot\|$. Let $\Delta = \{(t,s) \in \mathbb{R}_+^2 : t \geq s\}$ and $T = \{(t,s,r) \in \mathbb{R}_+^3 : t \geq s \geq r\}$.

We recall that an operator-valued function $\Phi \colon \Delta \to \mathcal{B}(X)$ is called an *evolution operator* on the Banach space X iff:

Received: May, 2012. *Revised:* July 20, 2012.
2010 *Mathematics Subject Classification*: 34D05, 34D09.
Key words and phrases: Evolution operator, uniform polynomial trichotomy.

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(e_1) $\Phi(t,t) = I$, for all $t \geq 0$;
(e_2) $\Phi(t,s)\Phi(s,r) = \Phi(t,r)$, for all $(t,s,r) \in T$.

An evolution operator $\Phi\colon \Delta \to \mathcal{B}(X)$ is called *strongly measurable* evolution operator, iff

(e_3) for every $(s,x) \in \mathbb{R}_+ \times X$ the mapping defined by $t \mapsto \| \Phi(t,s)x \|$ is measurable on $[s,\infty)$.

Definition 2.1. An application $P\colon \mathbb{R}_+ \to \mathcal{B}(X)$ is said to be *a projection family* on X if $P^2(t) = P(t)$, for all $t \geq 0$.

Definition 2.2. Three projection families $P_1, P_2, P_3\colon \mathbb{R}_+ \to \mathcal{B}(X)$ are said to be *compatible with the evolution operator* $\Phi\colon \Delta \to \mathcal{B}(X)$, if

(i) $P_1(t) + P_2(t) + P_3(t) = I$, for all $t \geq 0$;
(ii) $P_i(t)P_j(t) = 0$, for all $t \geq 0$, $i, j \in \{1,2,3\}$, $i \neq j$;
(iii) $\Phi(t,r)P_k(r) = P_k(t)\Phi(t,r)$, for all $(t,r) \in \Delta$ and all $k \in \{1,2,3\}$.

In what follows, when consider $\Phi\colon \Delta \to \mathcal{B}(X)$ an evolution operator on X and $P_1, P_2, P_3\colon \mathbb{R}_+ \to \mathcal{B}(X)$ projection families compatible with Φ, we will denote $\Phi_k(t,r) = \Phi(t,r)P_k(r)$, for all $(t,r) \in \Delta$ and all $k \in \{1,2,3\}$.

Definition 2.3. An evolution operator $\Phi : \Delta \to \mathcal{B}(X)$ is said to be *uniformly trichotomic* if there exists $N \geq 1$ such that:

$$\|\Phi_1(t,r)x\| \leq N\|\Phi_1(s,r)x\|, \quad (ut_1)$$

$$\|\Phi_2(s,r)x\| \leq N\|\Phi_2(t,r)x\|, \quad (ut_2)$$

$$\|\Phi_3(s,r)x\| \leq N\|\Phi_3(t,r)x\|, \quad (ut_3)$$

$$\|\Phi_3(t,r)x\| \leq N\|\Phi_3(s,r)x\|, \quad (ut_4)$$

for all $(t,s,r,x) \in T \times X$.

Remark 2.1. The evolution operator $\Phi\colon \Delta \to \mathcal{B}(X)$ is uniformly trichotomic if and only if there exists $N \geq 1$ such that:

(i) the evolution operator Φ_1 is uniformly stable;
(ii) the evolution operator Φ_2 is uniformly instable;
(iii) the evolution operator Φ_3 is uniformly stable and uniformly instable.

Definition 2.4. An evolution operator $\Phi\colon \Delta \to \mathcal{B}(X)$ is *uniformly exponentially trichotomic* if there are some constants $\alpha > 0$ and $N \geq 1$ such that:

$$\|\Phi_1(t,r)x\| \leq Ne^{-\alpha(t-s)}\|\Phi_1(s,r)x\|, \quad (uet_1)$$

$$\|\Phi_2(s,r)x\| \leq Ne^{-\alpha(t-s)}\|\Phi_2(t,r)x\|, \quad (uet_2)$$

$$\|\Phi_3(s,r)x\| \leq Ne^{\alpha(t-s)}\|\Phi_3(t,r)x\|, \quad (uet_3)$$

$$\|\Phi_3(t,r)x\| \leq Ne^{\alpha(t-s)}\|\Phi_3(s,r)x\|, \quad (uet_4)$$

for all $(t,s,r,x) \in T \times X$.

The constant $\alpha > 0$ is called *the exponential trichotomy constant* of Φ.

Definition 2.5. An evolution operator $\Phi\colon \Delta \to \mathcal{B}(X)$ is said to be *uniformly polynomially trichotomic* if there are some constants $N \geq 1$ and $\alpha > 1$ such that:

$$\|\Phi_1(t,r)x\| \leq N\left(\tfrac{s+1}{t+1}\right)^{\alpha}\|\Phi_1(s,r)x\|, \quad (upt_1)$$

$$\|\Phi_2(s,r)x\| \leq N\left(\tfrac{s+1}{t+1}\right)^{\alpha}\|\Phi_2(t,r)x\|, \quad (upt_2)$$

$$\|\Phi_3(s,r)x\| \leq N\left(\tfrac{t+1}{s+1}\right)^{\alpha}\|\Phi_3(t,r)x\|, \quad (upt_3)$$

$$\|\Phi_3(t,r)x\| \leq N\left(\tfrac{t+1}{s+1}\right)^{\alpha}\|\Phi_3(s,r)x\|, \quad (upt_4)$$

for all $(t,s,r,x) \in T \times X$.

The constant $\alpha > 1$ is called *the polynomial trichotomy constant of* Φ.

Remark 2.2. (i) For $P_3 = 0$, we obtain the property of *uniform polynomial dichotomy.*
(ii) For $P_2 = P_3 = 0$, we obtain the property of *uniform polynomial stability.*
(iii) For $P_1 = P_3 = 0$, we obtain the property of *uniform polynomial instability.*

Proposition 2.1. *If the uniform trichotomic evolution operator $\Phi\colon \Delta \to \mathcal{B}(X)$ is uniformly exponentially trichotomic with the exponential trichotomy constant $\alpha > 1$, then it is uniformly polynomially trichotomic.*

Proof. According to the hypothesis there exist $N \geq 1$ and $\alpha > 1$ such that the conditions (ut_1)-(ut_4) and (uet_1)-(uet_4) hold. Using the inequality

$$\frac{e^s}{s+1} \leq \frac{e^t}{t+1},$$

for $t \geq s \geq 0$, we have

$$\|\Phi_1(t,r)x\| \leq Ne^{-\alpha(t-s)}\|\Phi_1(s,r)x\| \leq N\left(\frac{s+1}{t+1}\right)^{\alpha}\|\Phi_1(s,r)x\|$$

$$\|\Phi_2(s,r)x\| \leq Ne^{-\alpha(t-s)}\|\Phi_2(t,r)x\| \leq N\left(\frac{s+1}{t+1}\right)^{\alpha}\|\Phi_2(t,r)x\|$$

$$\|\Phi_3(s,r)x\| \leq N\|\Phi_3(t,r)x\| \leq N\left(\frac{t+1}{s+1}\right)^{\alpha}\|\Phi_3(t,r)x\|$$

and

$$\|\Phi_3(t,r)x\| \leq N\|\Phi_3(s,r)x\| \leq N\left(\frac{t+1}{s+1}\right)^{\alpha}\|\Phi_3(s,r)x\|,$$

for all $(t,s,r,x) \in T \times X$, which ends the proof. □

The converse of the preceding proposition is not valid. This fact is illustrated by the following example.

Example 2.1. Let $X = \mathbb{R}^3$. The application $\Phi\colon \Delta \to \mathcal{B}(X)$,

$$\Phi(t,s)x = \left(\frac{s^2+1}{t^2+1}x_1, \frac{t^2+1}{s^2+1}x_2, \left(\frac{t+1}{s+1}\right)^2 x_3\right)$$

is an evolution operator, where $t \geq s \geq 0$ and $x \in X$.

We consider the projections $P_1, P_2, P_3 \colon \mathbb{R}_+ \to \mathcal{B}(X)$, $P_1(s)x = (x_1, 0, 0)$, $P_2(s) = (0, x_2, 0)$, $P_3(s) = (0, 0, x_3)$, for all $s \geq 0$ and all $x = (x_1, x_2, x_3) \in X$, compatible with the evolution operator Φ. We have that:

$$(t+1)^2 \parallel \Phi_1(t,s)x \parallel = (t+1)^2 \frac{s^2+1}{t^2+1} \parallel P_1(s)x \parallel \leq 2(s+1)^2 \parallel P_1(s)x \parallel;$$
$$(t+1)^2 \parallel P_2(s)x \parallel \leq 2(s+1)^2 \parallel \Phi_2(t,s)x \parallel;$$
$$(s+1)^2 \|P_3(s)x\| \leq 2(t+1)^2 \|\Phi_3(t,s)x\|;$$
$$(s+1)^2 \|\Phi_3(t,s)x\| = (s+1)^2 \left(\frac{t+1}{s+1}\right)^2 \|P_3(s)x\| \leq 2(t+1)^2 \|P_3(s)x\|,$$

for all $(t, s, x) \in \Delta \times X$. Thus, Φ is uniformly polynomially trichotmic with the polynomial trichotomy constant $\alpha = 2$.

If we assume that Φ is uniformly exponentially trichotomic, then there exist $N \geq 1$ and $\alpha > 0$ such that

$$\frac{s^2+1}{t^2+1} e^{\alpha(t-s)} \leq N,$$

for every $(t, s) \in \Delta$. For $s = 0$ and $t \to \infty$, we obtain a contradiction and hence Φ is not uniformly exponentially trichotomic.

Remark 2.3. Let $\Phi \colon \Delta \to \mathcal{B}(X)$ be a uniform trichotomic evolution operator. Then Φ is uniformly polynomially trichotomic if and only if Φ_1 is uniformly polynomially stable and Φ_2 is uniformly polynomially instable.

3. THE MAIN RESULT

Definition 3.1. Three projection families $P_1, P_2, P_3 \colon \mathbb{R}_+ \to \mathcal{B}(X)$ are said to be *polynomial compatible with the evolution operator* $\Phi \colon \Delta \to \mathcal{B}(X)$, if

- (c_1) $P_1(t) + P_2(t) + P_3(t) = I$, for all $t \geq 0$;
- (c_2) $P_i(t)P_j(t) = 0$, for all $t \geq 0$, $i, j \in \{1, 2, 3\}$, $i \neq j$;
- (c_3) $\Phi(t, r)P_k(r) = P_k(t)\Phi(t, r)$, for all $(t, r) \in \Delta$ and all $k \in \{1, 2, 3\}$;
- (c_4) there exist $M \geq 1$

and $\omega > 0$ such that

$$\|\Phi_1(t,r)x\| \leq M \left(\frac{t+1}{s+1}\right)^{\omega} \|\Phi_1(s,r)x\| \quad \text{and} \quad \|\Phi_2(s,r)x\| \leq M \left(\frac{t+1}{s+1}\right)^{\omega} \|\Phi_2(t,r)x\|,$$

for all $(t, s, r, x) \in T \times X$.

Theorem 3.1. *Let $\Phi \colon \Delta \to \mathcal{B}(X)$ be a strongly measurable evolution operator and let P_1, P_2, P_3 be three projection families polynomial compatible with Φ. The evolution operator Φ is uniformly polynomially trichotomic if and only if there exist $D \geq 1$ and $d > 1$ such that:*

$$\text{(a)} \quad \int_s^t \frac{(\tau+1)^{d-1}}{(s+1)^d} \|\Phi_1(\tau,r)x\| d\tau \leq D\|\Phi_1(s,r)x\|;$$
$$\text{(b)} \quad \int_s^t \frac{(t+1)^d}{(\tau+1)^{d+1}} \|\Phi_2(\tau,r)x\| d\tau \leq D\|\Phi_2(t,r)x\|;$$

$$\text{(c)} \int_s^t \left(\frac{s+1}{\tau+1}\right)^d \|\Phi_3(\tau,r)x\|d\tau \le D\|\Phi_3(s,r)x\|;$$
$$\text{(d)} \int_s^t \left(\frac{\tau+1}{t+1}\right)^d \|\Phi_3(\tau,r)x\|d\tau \le D\|\Phi_3(t,r)x\|,$$

for all $(t,s,r,x) \in T \times X$.

Proof. Necessity.

Let $(t,s,x) \in \Delta \times X$ and $d \in (1,\alpha)$. We have that

$$\int_s^t \frac{(\tau+1)^{d-1}}{(s+1)^d} \parallel \Phi_1(\tau,r)x \parallel d\tau \le N(s+1)^{\alpha-d} \parallel \Phi_1(s,r)x \parallel \cdot$$
$$\cdot \int_s^t (\tau+1)^{d-\alpha-1}d\tau \le D\|\Phi_1(s,r)x\|$$

and

$$\int_s^t \frac{(t+1)^d}{(\tau+1)^{d+1}} \parallel \Phi_2(\tau,r)x \parallel d\tau \le N(t+1)^{d-\alpha} \parallel \Phi_2(t,r)x \parallel \cdot$$
$$\cdot \int_s^t (\tau+1)^{\alpha-d-1}d\tau \le D\|\Phi_2(t,r)x\|,$$

where $D = \dfrac{N}{\alpha-d}$.

Using (upt_3) and (upt_4) and integrating on $[s,t]$ we obtain (c) and (d).

Sufficiency. Let $(t,s,x) \in \Delta \times X$. If $t \ge 2s+1$ we have that

$$\left(\frac{t+1}{s+1}\right)^d \parallel \Phi_1(t,r)x \parallel = \frac{2}{t+1}\int_{\frac{t-1}{2}}^t \left(\frac{t+1}{s+1}\right)^d \|\Phi_1(t,r)x\|d\tau$$
$$= 2\int_{\frac{t-1}{2}}^t \frac{(\tau+1)^{d-1}}{(s+1)^d}\left(\frac{t+1}{\tau+1}\right)^{d-1} \parallel \Phi_1(t,\tau) \parallel \|\Phi_1(\tau,r)x\,d\tau$$
$$\le 2^{d+\omega}M\int_s^t \frac{(\tau+1)^{d-1}}{(s+1)^d} \parallel \Phi_1(\tau,s)x \parallel d\tau$$
$$\le 2^{d+\omega}MD \parallel \Phi_1(s,r)x \parallel .$$

Therefore

$$\parallel \Phi_1(t,s)x \parallel \le 2^{d+\omega}MD\left(\frac{s+1}{t+1}\right)^d \parallel \Phi_1(s,r)x \parallel . \tag{3.1}$$

If $t \in [s, 2s+1)$ we have that

$$\left(\frac{t+1}{s+1}\right)^d \parallel \Phi_1(t,s)x \parallel \le 2^{d+\omega}M \parallel \Phi_1(s,r)x \parallel . \tag{3.2}$$

Let $c = \displaystyle\int_1^2 \frac{1}{u^{d+1}}du$, then for $t \ge 2s+1$ we have that

$$c \parallel \Phi_2(s,r)x \parallel = (s+1)^d \int_s^{2s+1} \frac{1}{(\tau+1)^{d+1}} \parallel \Phi_2(s,r)x \parallel \, d\tau$$
$$\leq M2^\omega \left(\frac{s+1}{t+1}\right)^d \int_s^t \frac{(t+1)^d}{(\tau+1)^{d+1}} \parallel \Phi_2(\tau,r)x \parallel \, d\tau$$
$$\leq M2^\omega D \left(\frac{s+1}{t+1}\right)^d \parallel \Phi_2(t,r)x \parallel .$$

Therefore

$$\parallel \Phi_2(s,r)x \parallel \leq \frac{M2^\omega D}{c} \left(\frac{s+1}{t+1}\right)^d \parallel \Phi_2(t,r)x \parallel . \tag{3.3}$$

For $t \in [s, 2s+1)$ we have that

$$\left(\frac{t+1}{s+1}\right)^d \parallel \Phi_2(s,r)x \parallel \leq M2^{d+\omega} \parallel \Phi_2(t,r)x \parallel . \tag{3.4}$$

Using the relations (3.1), (3.2), (3.3) and (3.4), we obtain the conditions (upt_1) and (upt_2) from Definition 2.5. Relations (upt_3) and (upt_4) follow from (c) respectively (d). Using this fact we obtain that Φ is uniformly polynomially trichotomic. □

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