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OPTIMALITY CONDITIONS AND DUALITY IN MINMAX FRACTIONAL PROGRAMMING, PART I: NECESSARY AND SUFFICIENT OPTIMALITY CONDITIONS

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Abstract. The purpose of this paper is to develop a fairly large number of sets of global parametric sufficient optimality conditions under various generalized $(\mathcal{F}, b, \phi, \rho, \theta)$-univexity assumptions for a continuous minmax fractional programming problem involving arbitrary norms.

1. INTRODUCTION

In this paper, we present a set of necessary optimality conditions and a multitude of global parametric sufficient optimality results under a variety of generalized $(\mathcal{F}, b, \phi, \rho, \theta)$-univexity conditions for the following continuous minmax fractional programming problem involving nondifferentiable functions:

$$(P) \qquad \text{Minimize } \max_{y\in Y} \frac{f(x,y) + \|A(y)x\|_a}{g(x,y) - \|B(y)x\|_b}$$

subject to

$$G_j(x) + \|C_j x\|_{c(j)} \le 0,\ j \in \underline{q}, \quad H_k(x) = 0,\ k \in \underline{r}, \quad x \in X,$$

where X is a nonempty open convex subset of $\mathbb{R}^n$ (n-dimensional Euclidean space), Y is a compact metrizable topological space, $f(\cdot, y),\ g(\cdot, y),\ y \in Y,\ G_j,\ j \in \underline{q} \equiv \{1, 2, \ldots, q\}$, and $H_k,\ k \in \underline{r}$, are real-valued functions defined on X, for each $y \in Y$ and $j \in \underline{q},\ A(y),\ B(y)$, and C_j are respectively $\ell \times n,\ m \times n$, and $n_j \times n$ matrices, $\|\cdot\|_a,\ \|\cdot\|_b$, and $\|\cdot\|_{c(j)}$ are arbitrary norms on $\mathbb{R}^\ell,\ \mathbb{R}^m$, and $\mathbb{R}^{n_j}$, respectively, and for each $y \in Y,\ g(x,y) - \|B(y)x\|_b > 0$ for all x satisfying the constraints of (P).

Evidently, (P) is a general prototype optimization model that contains as special cases a very large number of standard, well-known mathematical programming problems some of which have been extensively investigated in the related literature. More specifically, the following collection of seventeen problems can be viewed as particular cases of (P):

$$(P1) \qquad \operatorname*{Minimize}_{x\in\mathbb{F}} \max_{y\in Y} f(x,y) + \|A(y)x\|_a,$$

where $\mathbb{F}$ (assumed to be nonempty) is the feasible set of (P), that is,

$$\mathbb{F} = \{x \in X : G_j(x) + \|C_j x\|_{c(j)} \le 0,\ j \in \underline{q}, \quad H_k(x) = 0, \ \ k \in \underline{r}\};$$

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$$(P2) \qquad \underset{x\in\mathbb{F}}{\text{Minimize}} \ \max_{y\in Y} \frac{f(x,y)}{g(x,y)};$$

$$(P3) \qquad \underset{x\in\mathbb{F}}{\text{Minimize}} \ \max_{y\in Y} \ f(x,y);$$

$$(P4) \qquad \underset{x\in\mathbb{F}}{\text{Minimize}} \ \max_{y\in Y} \frac{f(x,y)+\langle x,K(y)x\rangle^{\frac{1}{2}}}{g(x,y)-\langle x,L(y)x\rangle^{\frac{1}{2}}}$$

subject to

$$G_j(x)+\langle x,M_jx\rangle^{\frac{1}{2}}\le 0, \ \ j\in\underline{q}, \quad H_k(x)=0, \ \ k\in\underline{r}, \quad x\in X,$$

where f, g, G_j, $j\in\underline{q}$, H_k, $k\in\underline{r}$, and X are as defined in the description of (P), $K(y), L(y)$, $y\in Y$, and M_j, $j\in\underline{q}$, are $n\times n$ symmetric positive semidefinite matrices, and $\langle u,v\rangle$ denotes the inner (scalar) product of the ν-dimensional vectors u and v, that is, $\langle u,v\rangle=\sum_{i=1}^{\nu} u_iv_i$, where u_i and v_i are the ith components of u and v, respectively;

$$(P5) \qquad \underset{x\in\mathbb{G}}{\text{Minimize}} \ \max_{y\in Y} f(x,y)+\langle x,K(y)x\rangle^{\frac{1}{2}},$$

where $\mathbb{G}$ is the feasible set of $(P4)$, that is,

$$\mathbb{G}=\{x\in X: G_j(x)+\langle x,M_jx\rangle^{\frac{1}{2}}\le 0, \ j\in\underline{q}, \quad H_k(x)=0, \ \ k\in\underline{r}\};$$

$$(P6) \qquad \underset{x\in\mathbb{F}}{\text{Minimize}} \ \max_{1\le i\le p} \frac{f_i(x)+\|A_ix\|_a}{g_i(x)-\|B_ix\|_b};$$

$$(P7) \qquad \underset{x\in\mathbb{F}}{\text{Minimize}} \ \max_{1\le i\le p} f_i(x)+\|A_ix\|_a;$$

$$(P8) \qquad \underset{x\in\mathbb{F}}{\text{Minimize}} \ \frac{f_1(x)+\|A_1x\|_a}{g_1(x)-\|B_1x\|_b};$$

$$(P9) \qquad \underset{x\in\mathbb{F}}{\text{Minimize}} \ f_1(x)+\|A_1x\|_a;$$

$$(P10) \qquad \underset{x\in\mathbb{G}}{\text{Minimize}} \ \max_{1\le i\le p} \frac{f_i(x)+\langle x,K_ix\rangle^{\frac{1}{2}}}{g_i(x)-\langle x,L_ix\rangle^{\frac{1}{2}}},$$

where for each $i\in\underline{p}$, f_i and g_i are real-valued functions defined on X, and K_i and L_i are $n\times n$ symmetric positive semidefinite matrices;

$$(P11) \qquad \underset{x\in\mathbb{G}}{\text{Minimize}} \ \max_{1\le i\le p} f_i(x)+\langle x,K_ix\rangle^{\frac{1}{2}};$$

$$(P12) \qquad \underset{x\in\mathbb{G}}{\text{Minimize}} \ \frac{f_1(x)+\langle x,K_1x\rangle^{\frac{1}{2}}}{g_1(x)-\langle x,L_1x\rangle^{\frac{1}{2}}};$$

$$(P13) \qquad \underset{x\in\mathbb{G}}{\text{Minimize}} \ f_1(x)+\langle x,K_1x\rangle^{\frac{1}{2}};$$

$$(P14) \qquad \underset{x\in\mathbb{H}}{\text{Minimize}} \ \max_{1\le i\le p} \frac{f_i(x)}{g_i(x)};$$

$$(P15) \qquad \underset{x\in\mathbb{H}}{\text{Minimize}} \ \max_{1\le i\le p} \ f_i(x);$$

$$(P16) \qquad \underset{x\in\mathbb{H}}{\text{Minimize}} \ \frac{f_1(x)}{g_1(x)};$$

$$(P17) \qquad \underset{x\in\mathbb{H}}{\text{Minimize}} \ f_1(x),$$

where

$$\mathbb{H} = \{x \in X : G_j(x) \leq 0,\ j \in \underline{q}, \quad H_k(x) = 0,\ \ k \in \underline{r}\}.$$

The problem $(P4)$ is obtained from (P) by choosing $\|\cdot\|_a, \|\cdot\|_b, y \in Y$, and $\|\cdot\|_{c(j)}, j \in \underline{q}$, to be the ℓ_2-norm $\|\cdot\|_2$, and defining $K(y) = A(y)^T A(y), L(y) = B(y)^T B(y), y \in Y$, and $M_j = C_j^T C_j, j \in \underline{q}$. Similarly, $(P6)$ is obtained from (P) by setting $Y = \{1, 2, \ldots, p\}$, $f(x, i) = f_i(x)$, $g(x, i) = g_i(x)$, $A(i) = A_i$, and $B(i) = B_i,\ i \in \underline{p}$.

It appears that the basic continuous minmax programming problem $(P3)$ was investigated in a serious manner for the first time by Danskin [16] who established necessary optimality conditions for this problem and clearly demonstrated its utility in providing realistic models for some significant real-world problems. Following the pioneering work of Danskin, many authors have been studying various aspects of different types of continuous minmax programming problems in the past four decades. For a wealth of information about several facets of continuous minmax problems, including optimality conditions, duality relations, solution algorithms, and areas of applications, the reader is referred to [5] - [7], [10, 11, 17, 27], [34] - [37], [39, 44], [47] - [50], [54, 55], [57] - [61].

The discrete minmax optimization problems $(P6)$, $(P10)$, and $(P14)$ are known as *generalized fractional programming problems* in the literature of mathematical programming, and have been the focus of considerable attention in the past two decades. The salient features of these problems that appear to be responsible for this surge of interest are their capability to provide realistic models for some significant real-world problems (they have been encountered in multiobjective programming [1], approximation theory [2, 3, 33, 58], goal programming [12, 32], facility location planning [4], and economics [56], among others), their mathematical tractability (they can be transformed to equivalent parametric nonlinear programming problems with nonfractional objective functions), and their generality (they contain, as special cases, three important classes of nonlinear programming problems, namely, problems with fractional, discrete max, and conventional objective functions).

The notion of duality for generalized linear fractional programming was initially considered by von Neumann [56] in the context of an economic equilibrium problem. More recently, a number of optimality criteria, duality results, and computational algorithms for several classes of generalized linear and nonlinear fractional programming problems have appeared in the related literature. A fairly extensive list of references pertaining to various aspects of these problems is given in [63].

Perhaps, one of the most extensively studied and widely used special cases of (P) is the basic fractional programming model $(P16)$. Various aspects of this popular optimization model have been investigated vigorously in the past four decades culminating in the accumulation of a relatively large literature. One of the principal reasons for such a continual immense interest in this particular optimization paradigm appears to be its capability in providing realistically representative models for a number of significant classes of problems in the fields of operations research, management science, and economics. This feature is primarily due to the fact that in many areas, including resource allocation, transportation, production planning, sequencing, inventory control, financial planning, maintenance and replacement scheduling, and reliability assessment, ratios such as profit/capital,

profit/revenue, return/cost, return/risk, cost/time, profit/time, etc., can serve as useful measures of system performance. Proper characterization and utilization of these measures often requires optimization of certain ratios which, in turn, gives rise to the formulation of fractional programming problems. In noneconomic situations, fractional programming problems have arisen in information theory, stochastic programming, numerical analysis, approximation theory, cluster analysis, graph theory, multifacility location theory, decomposition of large-scale mathematical programming problems, and goal programming, among others. For comprehensive surveys and extensive lists of references dealing with several aspects of fractional programming, including modeling properties, actual and potential areas of applications, optimality conditions, duality formulations, sensitivity and stability analysis, and computational algorithms, the reader is referred to [14, 15, 46, 51, 52, 53].

The rest of this paper is organized as follows. In Section 2, we recall a number of definitions and auxiliary results which will be needed in the sequel. In Section 3, we combine a Dinkelbach-type indirect parametric approach with some nonsmooth analytic techniques to establish a set of necessary optimality conditions for (P). In Section 4, we begin our discussion of sufficient optimality conditions where we formulate and prove numerous sets of sufficiency criteria under a variety of generalized $(\mathcal{F}, b, \phi, \rho, \theta)$-univexity assumptions that are placed on the individual as well as certain combinations of the problem functions. Utilizing two partitioning schemes, in Section 5 we establish several sets of generalized sufficiency results each of which is in fact a family of such results whose members can easily be identified by appropriate choices of certain sets and functions. Finally, in Section 6 we summarize our main results and also point out some further research opportunities arising from certain modifications of the principal problem model considered in this paper.

It is evident that the necessary optimality conditions as well as all the sufficient optimality results obtained for (P) are also applicable, when appropriately specialized, to each one of the seventeen classes of optimization problems designated above as $(P1)$ - $(P17)$, which are particular cases of (P). Since in most cases these results can easily be modified and restated for each one of the seventeen problems, we shall not state them explicitly.

Optimization problems containing norms arise naturally in many areas of the decision sciences, applied mathematics, and engineering. They are encountered most frequently in facility location problems, approximation theory, and engineering design. A number of these problems have already been investigated in the related literature. Likewise, optimization problems involving square roots of positive semidefinite quadratic forms have arisen in stochastic programming, multifacility location problems, and portfolio selection problems, among others. A fairly extensive list of references pertaining to several aspects of these two classes of problems is given in [62].

2. PRELIMINARIES

For the purpose of formulating and proving various collections of generalized sufficiency criteria for (P), in this study we shall use a new class of generalized convex functions, called $(\mathcal{F}, b, \phi, \rho, \theta)$**-univex functions**, which will be defined later in this section. This class of functions may be viewed as a combination of several previously defined types of generalized convex functions. Its main ingredients are invex functions, (η, ρ)-invex functions, $\mathcal{F}$-convex functions, and univex functions which were introduced in [22], [28], [23], and [8], respectively. These functions were proposed as generalizations of the class of differentiable convex functions.

Prior to giving the definitions of the new classes of generalized convex functions, it will be useful for purposes of reference and comparison to recall the definitions of the principal components of these functions mentioned above. We shall keep this review to a bare minimum because our primary objective is only to put a number of interrelated generalized convexity concepts into proper perspective. For this reason, we shall only reproduce the essential forms of the definitions without elaborating on their refinements, variants, special cases, and other manifestations. For full discussions of the consequences and applications of the underlying ideas, the reader may consult the original sources. We begin by defining an invex function which occupies a pivotal position in a vast array of generalized convex functions some of which are specified in the following definitions.

Definition 2.1. Let f be a differentiable real-valued function defined on the open convex set X. Then f is said to be η-*invex* (invex with respect to η) at y if there exists a function $\eta\colon X \times X \to \mathbb{R}^n$ such that for each $x \in X$,

$$f(x) - f(y) \geq \langle \nabla f(y), \eta(x, y) \rangle,$$

where $\nabla f(y) = (\partial f(y)/\partial y_1, \partial f(y)/\partial y_2, \ldots, \partial f(y)/\partial y_n)$ is the gradient of f at y; f is said to be η-*invex* on X if the above inequality holds for all $x, y \in X$.

From this definition it is clear that every differentiable real-valued convex function is invex with $\eta(x, y) = x - y$. This generalization of the concept of convexity was originally proposed by Hanson [22] who showed that for a nonlinear programming problem of the form

$$\text{Minimize } f(x) \text{ subject to } g_i(x) \leq 0,\ i \in \underline{m},\ x \in \mathbb{R}^n,$$

where the differentiable functions $f, g_i\colon \mathbb{R}^n \to \mathbb{R}$, $i \in \underline{m}$, are invex with respect to the same function $\eta\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, the Karush-Kuhn-Tucker necessary optimality conditions are also sufficient. The term *invex* (for *in*variant con*vex*) was coined by Craven [13] to signify the fact that the invexity property, unlike convexity, remains invariant under bijective coordinate transformations.

In a similar manner, one can readily define η-pseudoinvex and η-quasiinvex functions as generalizations of differentiable pseudoconvex and quasiconvex functions.

The notion of invexity has been generalized in several directions. For our present purposes, we shall need a simple extension of invexity, namely, ρ-invexity which was originally defined in [28].

Let η be a function from $X \times X$ to $\mathbb{R}^n$, and let h be a differentiable real-valued function defined on X.

Definition 2.2. The function h is said to be (η, ρ)-*invex* at x^* if there exists $\rho \in \mathbb{R}$ such that for each $x \in X$,

$$h(x) - h(x^*) \geq \langle \nabla h(x^*), \eta(x, x^*) \rangle + \rho \|x - x^*\|^2.$$

Definition 2.3. The function h is said to be (η, ρ)-*pseudoinvex* at $x^* \in X$ if there exists $\rho \in \mathbb{R}$ such that for each $x \in X$,

$$\langle \nabla h(x^*), \eta(x, x^*) \rangle \geq -\rho \|x - x^*\|^2 \ \Rightarrow\ h(x) \geq h(x^*).$$

Definition 2.4. The function h is said to be (η, ρ)-*quasiinvex* at $x^* \in X$ if there exists $\rho \in \mathbb{R}$ such that for each $x \in X$,

$$h(x) \leq h(x^*) \ \Rightarrow\ \langle \nabla h(x^*), \eta(x, x^*) \rangle \leq -\rho \|x - x^*\|^2.$$

The notion of invexity has been extended in many ways. For more information pertaining to various generalizations of invex functions and their applications along with extensive lists of relevant references, the reader is referred to [9, 19, 20, 29, 38, 40, 42, 45].

Another generalization of an invex function is an $\mathcal{F}$-convex function which is defined in terms of a sublinear function, that is, a function that is subadditive and positively homogeneous.

Definition 2.5. A function $\mathcal{F}\colon X \to \mathbb{R}$ is said to be *sublinear (superlinear)* if $\mathcal{F}(x+y) \leq (\geq)\mathcal{F}(x) + \mathcal{F}(y)$ for all $x, y \in X$, and $\mathcal{F}(ax) = a\mathcal{F}(x)$ for all $x \in X$ and $a \in \mathbb{R}_+ \equiv [0, \infty)$.

Now combining the definitions of $\mathcal{F}$-convex and (η, ρ)-invex functions given in [23] and [28], respectively, we can define $(\mathcal{F}, \rho)$-convex, $(\mathcal{F}, \rho)$-pseudoconvex, and $(\mathcal{F}, \rho)$-quasiconvex functions.

Let g be a real-valued differentiable function defined on X, and assume that for each $x, y \in X$, the function $\mathcal{F}(x, y; \cdot)\colon X \to \mathbb{R}$ is sublinear.

Definition 2.6. The function g is said to be $(\mathcal{F}, \rho)$-*convex* at y if there exists a real number ρ such that for each $x \in X$,

$$g(x) - g(y) \geq \mathcal{F}(x, y; \nabla g(y)) + \rho\|x - y\|^2.$$

Definition 2.7. The function g is said to be $(\mathcal{F}, \rho)$-*pseudoconvex* at y if there exists a real number ρ such that for each $x \in X$,

$$\mathcal{F}(x, y; \nabla g(y)) \geq -\rho\|x - y\|^2 \;\;\Rightarrow\;\; g(x) \geq g(y).$$

Definition 2.8. The function g is said to be $(\mathcal{F}, \rho)$-*quasiconvex* at y if there exists a real number ρ such that for each $x \in X$,

$$g(x) \leq g(y) \;\;\Rightarrow\;\; \mathcal{F}(x, y; \nabla g(y)) \leq -\rho\|x - y\|^2.$$

Evidently, if in Definitions 2.6 - 2.8 we choose $\mathcal{F}(x, y; \nabla g(y)) = \langle \nabla g(y), \eta(x, y)\rangle$, where $\eta\colon X \times X \to \mathbb{R}^n$ is a given function, and set $\rho = 0$, then we see that they reduce to the definitions of η-invexity, η-pseudoinvexity, and η-quasiinvexity for the function g.

The foregoing classes of generalized convex functions have been utilized for establishing numerous sets of sufficient optimality conditions and a variety of duality results for several categories of static and dynamic optimization problems. For recent surveys and syntheses of these results, the reader is referred to [31, 43].

Another significant generalization of the notion of invexity, called *univexity*, which subsumes a number of previously proposed types of generalized convex functions, was recently given in [8]. We recall the definitions of univex, pseudounivex, and quasiunivex functions.

Let h be a differentiable real-valued function defined on X, let η be a function from $X \times X$ to $\mathbb{R}^n$, let Φ be a real-valued function defined on X, and let b be a function from $X \times X$ to $\mathbb{R}_+ \setminus \{0\} \equiv (0, \infty)$.

Definition 2.9. The function h is said to be *univex* at y with respect to η, Φ, and b if for each $x \in X$,

$$b(x, y)\Phi(h(x) - h(y)) \geq \langle \nabla h(y), \eta(x, y)\rangle.$$

Definition 2.10. The function h is said to be *pseudounivex* at y with respect to η, Φ, and b if for each $x \in X$,

$$\langle \nabla h(y), \eta(x, y)\rangle \geq 0 \;\;\Rightarrow\;\; b(x, y)\Phi(h(x) - h(y)) \geq 0.$$

Definition 2.11. The function h is said to be *quasiunivex* at y with respect to η, Φ, and b if for each $x \in X$,

$$b(x,y)\Phi(h(x)-h(y)) \le 0 \;\Rightarrow\; \langle \nabla h(y), \eta(x,y)\rangle \le 0.$$

Finally, we are in a position to give our definitions of generalized $(\mathcal{F}, b, \phi, \rho, \theta)$-univex functions. They are formulated by combining Definitions 2.5 - 2.11.

Let $x^* \in X$ and assume that the function $F\colon X \to \mathbb{R}$ is differentiable at x^*.

Definition 2.12. The function F is called *(strictly)* $(\mathcal{F}, b, \phi, \rho, \theta)$*-univex* at x^* if there exist functions $b\colon X \times X \to \mathbb{R}_+\backslash\{0\} \equiv (0,\infty)$, $\phi\colon \mathbb{R} \to \mathbb{R}$, $\rho\colon X \times X \to \mathbb{R}$, and $\theta\colon X \times X \to \mathbb{R}^n$, and a sublinear function $\mathcal{F}(x, x^*; \cdot)\colon \mathbb{R}^n \to \mathbb{R}$ such that for each $x \in X (x \ne x^*)$,

$$\phi\big(F(x) - F(x^*)\big)(>) \ge \mathcal{F}\big(x, x^*; b(x,x^*)\nabla F(x^*)\big) + \rho(x,x^*)\|\theta(x,x^*)\|^2.$$

Definition 2.13. The function F is said to be *(strictly)* $(\mathcal{F}, b, \phi, \rho, \theta)$*-pseudounivex* at x^* if there exist functions $b\colon X \times X \to \mathbb{R}_+\backslash\{0\} \equiv (0,\infty)$, $\phi\colon \mathbb{R} \to \mathbb{R}$, $\rho\colon X \times X \to \mathbb{R}$, and $\theta\colon X \times X \to \mathbb{R}^n$, and a sublinear function $\mathcal{F}(x, x^*; \cdot)\colon \mathbb{R}^n \to \mathbb{R}$ such that for each $x \in X (x \ne x^*)$,

$$\mathcal{F}\big(x, x^*; b(x,x^*)\nabla F(x^*)\big) \ge -\rho(x,x^*)\|\theta(x,x^*)\|^2 \;\Rightarrow\; \phi\big(F(x) - F(x^*)\big)(>) \ge 0.$$

Definition 2.14. The function F is said to be *(prestrictly)*$(\mathcal{F}, b, \phi, \rho, \theta)$*-quasiunivex* at x^* if there exist functions $b\colon X \times X \to \mathbb{R}_+\backslash\{0\} \equiv (0,\infty)$, $\phi\colon \mathbb{R} \to \mathbb{R}$, $\rho\colon X \times X \to \mathbb{R}$, and $\theta\colon X \times X \to \mathbb{R}^n$, and a sublinear function $\mathcal{F}(x, x^*; \cdot)\colon \mathbb{R}^n \to \mathbb{R}$ such that for each $x \in X$,

$$\phi\big(F(x) - F(x^*)\big)(<) \le 0 \;\Rightarrow\; \mathcal{F}\big(x, x^*; b(x,x^*)\nabla F(x^*)\big) \le -\rho(x,x^*)\|\theta(x,x^*)\|^2.$$

From the above definitions it is clear that if F is $(\mathcal{F}, b, \phi, \rho, \theta)$-univex at x^*, then it is both $(\mathcal{F}, b, \phi, \rho, \theta)$-pseudounivex and $(\mathcal{F}, b, \phi, \rho, \theta)$-quasiunivex at x^*, if F is $(\mathcal{F}, b, \phi, \rho, \theta)$-quasiunivex at x^*, then it is prestrictly $(\mathcal{F}, b, \phi, \rho, \theta)$-quasiunivex at x^*, and if F is strictly $(\mathcal{F}, b, \phi, \rho, \theta)$-pseudounivex at x^*, then it is $(\mathcal{F}, b, \phi, \rho, \theta)$-quasiunivex at x^*.

In the proofs of the sufficiency theorems, sometimes it may be more convenient to use certain alternative but equivalent forms of the above definitions. These are obtained by considering the contrapositive statements. For example, $(\mathcal{F}, b, \phi, \rho, \theta)$-quasiunivexity can be defined in the following equivalent way:

F is said to be $(\mathcal{F}, b, \phi, \rho, \theta)$-quasiunivex at x^* if there exist functions $b\colon X \times X \to \mathbb{R}_+\backslash\{0\} \equiv (0,\infty)$, $\phi\colon \mathbb{R} \to \mathbb{R}$, $\rho\colon X \times X \to \mathbb{R}$, and $\theta\colon X \times X \to \mathbb{R}^n$, and a sublinear function $\mathcal{F}(x, x^*; \cdot)\colon \mathbb{R}^n \to \mathbb{R}$ such that for each $x \in X$,

$$\mathcal{F}\big(x, x^*; b(x,x^*)\nabla F(x^*)\big) > -\rho(x,x^*)\|\theta(x,x^*)\|^2 \;\Rightarrow\; \phi\big(F(x) - F(x^*)\big) > 0.$$

Needless to say that the new classes of generalized convex functions specified in Definitions 2.12 - 2.14 contain a great variety of special cases that can easily be identified by appropriate choices of the functions $\mathcal{F}, b, \phi, \rho$, and θ. In particular, they subsume all the generalized convex functions specified in Definitions 2.1 - 2.11.

3. NECESSARY OPTIMALITY CONDITIONS

In this section, we establish a set of necessary optimality conditions for (P). We derive these conditions by combining the optimality results of [24] for a nonsmooth nonlinear programming problem with a Dinkelbach-type [18] indirect parametric approach. The auxiliary problem which makes this approach possible has the form

$$(P\lambda) \qquad \underset{x\in\mathbb{F}}{\text{Minimize}} \ \max_{y\in Y} \ \{f(x,y) + \|A(y)x\|_a - \lambda[g(x,y) - \|B(y)x\|_b]\},$$

where λ is a parameter. The relationships between this problem and (P) that are useful for our present purposes are stated in the following lemma whose proof is straightforward and hence omitted.

Lemma 3.1. *Let λ^* be the optimal value of (P), and let $v(\lambda)$ be the optimal value of $(P\lambda)$ for any fixed $\lambda \in \mathbb{R}$ such that $(P\lambda)$ has an optimal solution. Then the following statements are valid:*

(a) *If x^* is an optimal solution of (P), then it is an optimal solution of $(P\lambda^*)$ and $v(\lambda^*) = 0$.*

(b) *If $(P\bar{\lambda})$ has an optimal solution $\bar{x}$ for some $\bar{\lambda} \in \mathbb{R}$ with $v(\bar{\lambda}) = 0$, then $\bar{x}$ is an optimal solution of (P) and $\bar{\lambda} = \lambda^*$.*

Throughout this paper, the gradient and the subdifferential are always taken with respect to the first variable x. So $\nabla f(x,y)$ and $\partial(\|A(y)x\|)$ are the abbreviations for $\nabla_x f(x,y)$ and $\partial_x(\|A(y)x\|)$, respectively. We also posit the following assumptions which will remain in effect throughout the sequel:

(A1) For each $x \in X$, the functions $f(x,\cdot)$ and $g(x,\cdot)$ are continuous on Y.

(A2) Each entry of the matrices $A(y)$ and $B(y)$ is a continuous real-valued function defined on Y.

The following auxiliary result will be needed in the proof of the necessary optimality conditions for (P).

Lemma 3.2. [21] *Let Q be a metrizable compact topological space, and for each $\alpha \in Q$, let f_α be a real-valued Lipschitz function defined on $\mathbb{R}^n$ with the same Lipschitz constant. Let $x^* \in \mathbb{R}^n$ and assume that the mapping $\alpha \to f_\alpha(x^*)$ is continuous, and the set-valued mapping $(\alpha, x) \to \partial f_\alpha(x)$ is upper semicontinuous at (α, x^*) for all $\alpha \in Q$. Then*

$$\partial f(x^*) \subset co\big(\bigcup_{\alpha\in Q_0(x^*)} \partial f_\alpha(x^*)\big),$$

where $f(x) = \max\limits_{\alpha\in Q} f_\alpha(x)$, $Q_0(x) = \{\alpha \in Q : f_\alpha(x) = \max\limits_{\alpha\in Q} f_\alpha(x)\}$, and $co(S)$ indicates the convex hull of the set S.

We also need a formula for the subdifferential of a norm function.

Lemma 3.3. [25] *Let the function $f\colon \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x) = \|Ax\|_a$, where A is an $m \times n$ matrix and $\|\cdot\|_a$ is an arbitrary norm. Then the subdifferential $\partial f(x^*)$ of the convex function f at x^* is given by*

$$\partial f(x^*) = \{\zeta^T A : \|\zeta\|_a^* \le 1, \ \langle \zeta, Ax^*\rangle = \|Ax^*\|_a\},$$

where $\|\cdot\|_a^*$ is the dual to the norm $\|\cdot\|_a$, that is, $\|\delta\|_a^* = \max\limits_{\|\xi\|_a=1} |\langle \delta, \xi\rangle|$.

Using the formula above, we can prove the following lemma.

Lemma 3.4. *Let $F(x,y) = \|A(y)x\|_a$. Then the set-valued map $(x,y) \to \partial F(x,y)$ is upper semicontinuous.*

Proof. Suppose to the contrary that the statement is not true. Then there exist an $\epsilon > 0$ and a sequence $(x_n, y_n) \to (\bar{x}, \bar{y})$ such that $\partial(\|A(y_n)x_n\|_a)$ is not a subset of $\partial(\|A(\bar{y})\bar{x}\|_a) + \epsilon B$, where B is the unit ball in $\mathbb{R}^n$. By Lemma 3.3, there exists $\xi_n \in \mathbb{R}^\ell$ with $\|\xi_n\|_a^* \leq 1$ and $\|A(y_n)x_n\|_a = \langle \xi_n, A(y_n)x_n\rangle$ such that $\xi_n^T A(y_n) \notin \partial(\|A(\bar{y})\bar{x}\|_a + \epsilon B$. Without loss of generality, we may assume that $\xi_n \to \xi_0$. Since each entry of $A(y)$ is continuous, we conclude that $\xi_n^T A(y_n) \to \xi_0^T A(\bar{y})$, $\|\xi_0\|_a^* \leq 1$, and $\|A(\bar{y})\bar{x}\|_a = \langle \xi_0, A(\bar{y})\bar{x}\rangle$. Therefore, $\xi_0^T A(\bar{y}) \in \partial(\|A(\bar{y})\bar{x}\|_a)$, which contradicts the fact that $\xi_n^T A(y_n) \notin \partial(\|A(\bar{y})\bar{x}\|_a) + \epsilon B$. Therefore, the set-valued map $(x, y) \to \partial F(x, y)$ is upper semicontinuous. □

For $x \in X$, let

$$\bar{Y}(x) = \left\{ y \in Y : \frac{f(x,y) + \|A(y)x\|_a}{g(x,y) - \|B(y)x\|_b} = \max_{z \in Y} \frac{f(x,z) + \|A(z)x\|_a}{g(x,z) - \|B(z)x\|_b} \right\},$$

and

$$\mathbb{K}(x) = \Big\{ (p, \bar{y}, u, \alpha, \beta) : 1 \leq p \leq n+1;\ \bar{y} = (y^1, y^2, \ldots, y^p), y^i \in \bar{Y}(x);\ u \in \mathbb{R}_+^p,$$
$$\sum_{i=1}^{p} u_i = 1;\ \alpha = (\alpha^1, \alpha^2, \ldots, \alpha^p), \alpha^i \in \mathbb{R}^\ell;\ \beta = (\beta^1, \beta^2, \ldots, \beta^p),\ \beta^i \in \mathbb{R}^m \Big\}.$$

Next, with the help of Lemmas 3.1, 3.3, and 3.4, and the optimality results of [24] we prove the following theorem which is the main result of this section.

Theorem 3.1. *Let x^* be an optimal solution of (P). Assume that the functions $f(\cdot, y)$, $g(\cdot, y)$, $y \in Y$, G_j, $j \in \underline{q}$, and H_k, $k \in \underline{r}$, are continuously differentiable at x^*, and that any one of the constraint qualifications specified in* [24, 30] *holds at x^*. Then there exist $\lambda^* \in \mathbb{R}$, $(p^*, \bar{y}^*, u^*, \alpha^*, \beta^*) \in \mathbb{K}(x^*)$, $v^* \in \mathbb{R}_+^q \equiv \{v \in \mathbb{R}^q : v \geq 0\}$, $w^* \in \mathbb{R}^r$, and $\gamma^{*j} \in \mathbb{R}^{n_j}$, $j \in \underline{q}$, such that*

$$\sum_{i=1}^{p^*} u_i^* \{\nabla f(x^*, y^{*i}) + A(y^{*i})^T \alpha^{*i} - \lambda^* [\nabla g(x^*, y^{*i}) - B(y^{*i})^T \beta^{*i}]\}$$
$$+ \sum_{j=1}^{q} v_j^* [\nabla G_j(x^*) + C_j^T \gamma^{*j}] + \sum_{k=1}^{r} w_k^* \nabla H_k(x^*) = 0, \quad (3.1)$$

$$v_j^* [G_j(x^*) + \|C_j x^*\|_{c(j)}] = 0, \quad j \in \underline{q}, \quad (3.2)$$

$$\|\alpha^{*i}\|_a^* \leq 1, \quad \|\beta^{*i}\|_b^* \leq 1, \quad i \in \underline{p^*}, \quad (3.3)$$

$$\|\gamma^{*j}\|_{c(j)}^* \leq 1, \quad j \in \underline{q}, \quad (3.4)$$

$$\langle \alpha^{*i}, A(y^{*i})x^* \rangle = \|A(y^{*i})x^*\|_a, \quad \langle \beta^{*i}, B(y^{*i})x^* \rangle = \|B(y^{*i})x^*\|_b, \quad i \in \underline{p^*}, \quad (3.5)$$

$$\langle \gamma^{*j}, C_j x^* \rangle = \|C_j x^*\|_{c(j)}, \quad j \in \underline{q}. \quad (3.6)$$

Proof. Since x^* is an optimal solution of (P), by Lemma 3.1, it is an optimal solution of $(P\lambda^*)$, where λ^* is the optimal value of (P) at x^*. By Theorem 3.3 in [24], the Lagrange

multiplier set of the problem $(P\lambda^*)$ is not empty. Therefore, there exist $v^* \in \mathbb{R}^q_+$ and $w^* \in \mathbb{R}^r$, such that

$$0 \in \partial\Big\{\max_{y\in Y}\{f(x^*,y) + \|A(y)x^*\|_a - \lambda^*[g(x^*,y) - \|B(y)x^*\|_b]\}\Big\} + \sum_{j=1}^{q} v_j^* \partial(G_j(x^*) + \|C_j x\|) + \sum_{k=1}^{r} w_k^* \nabla H_k(x^*),$$

and (3.2) holds. By Lemmas 3.2 and 3.4, we know that

$$\partial\Big\{\max_{y\in Y}\{f(x^*,y) + \|A(y)x^*\|_a - \lambda^*[g(x^*,y) - \|B(y)x^*\|_b]\}\Big\} \subset co\Big\{\bigcup_{y\in Y_0(x^*)} \partial\big(f(x^*,y) + \|A(y)x^*\|_a - \lambda^*[g(x^*,y) - \|B(y)x^*\|_b]\big)\Big\}, \quad (3.7)$$

with

$$Y_0(x^*) = \{\bar{y} \in Y : D_{\lambda^*}(x^*,\bar{y}) = \max_{y\in Y} D_{\lambda^*}(x^*,y)\},$$

where

$$D_\lambda(x,y) = f(x,y) + \|A(y)x\|_a - \lambda[g(x,y) - \|B(y)x\|_b].$$

From Lemma 3.1 it is easy to see that $Y_0(x^*) = \bar{Y}(x^*)$. By Carathéodory's theorem [25] and Lemma 3.3, there exist $(p^*, \bar{y}^*, u^*, \alpha^*, \beta^*) \in \mathbb{K}(x^*)$ and $\gamma^{*j} \in \mathbb{R}^{n_j}$, $j \in \underline{q}$, as specified in the statement of the theorem, such that (3.1) - (3.6) hold. □

To give the reader a glimpse of what is involved in the process of modifying and restating the results of this paper for $(P1)$ - $(P17)$, we next specialize Theorem 3.1 for $(P6)$.

Theorem 3.2. *Let x^* be an optimal solution of $(P6)$ and assume that the functions f_i, g_i, $i \in \underline{p}$, G_j, $j \in \underline{q}$, and H_k, $k \in \underline{r}$, are continuously differentiable at x^*, and that any one of the constraint qualifications specified in* [24, 30] *holds at x^*. Then there exist $\lambda^* \in \mathbb{R}$, $u^* \in U \equiv \{u \in \mathbb{R}^p : u \geq 0,\ \sum_{i=1}^p u_i = 1\}$, $v^* \in \mathbb{R}^q_+$, $w^* \in \mathbb{R}^r$, $\alpha^{*i} \in \mathbb{R}^{\ell_i}$, $\beta^{*i} \in \mathbb{R}^{m_i}$, $i \in \underline{p}$, and $\gamma^{*j} \in \mathbb{R}^{n_j}$, $j \in \underline{q}$, such that*

$$\sum_{i=1}^{p} u_i^*\{\nabla f_i(x^*) + A_i^T\alpha^{*i} - \lambda^*[\nabla g_i(x^*) - B_i^T\beta^{*i}]\} + \sum_{j=1}^{q} v_j^*[\nabla G_j(x^*) + C_j^T\gamma^{*j}] + \sum_{k=1}^{r} w_k^*\nabla H_k(x^*) = 0,$$

$$u_i^*\{f_i(x^*) + \|A_ix^*\|_a - \lambda^*[g_i(x^*) - \|B_ix^*\|_b]\} = 0, \quad i \in \underline{p},$$

$$v_j^*[G_j(x^*) + \|C_jx^*\|_{c(j)}] = 0, \quad j \in \underline{q},$$

$$\|\alpha^{*i}\|_a^* \leq 1, \quad \|\beta^{*i}\|_b^* \leq 1, \quad i \in \underline{p},$$

$$\|\gamma^{*j}\|_{c(j)}^* \leq 1, \quad j \in \underline{q},$$

$$\langle\alpha^{*i}, A_ix^*\rangle = \|A_ix^*\|_a, \quad \langle\beta^{*i}, B_ix^*\rangle = \|B_ix^*\|_b, \quad i \in \underline{p},$$

$$\langle\gamma^{*j}, C_jx^*\rangle = \|C_jx^*\|_{c(j)}, \quad j \in \underline{q}.$$

The form and contents of the necessary optimality conditions given in Theorem 3.1 provide clear guidelines for formulating numerous sets of sufficient optimality conditions and many duality models for (P). The rest of this paper is devoted to investigating various sets of sufficiency criteria for (P). The sufficiency results established here may be viewed as substantial extensions and improvements of those obtained for $(P4)$ in [64]. A vast number of duality results for (P) which are based on Theorem 3.1 and the ensuing sufficiency results are discussed in [65, 66].

4. SUFFICIENT OPTIMALITY CONDITIONS

In this section, we present a number of sufficiency results in which various generalized $(\mathcal{F}, b, \phi, \rho, \theta)$-univexity assumptions are imposed on the individual as well certain combinations of the problem functions. In formulating these results, we shall use a streamlined version of the necessary conditions given in Theorem 3.1. Specifically, we use an altered version of these conditions obtained by dropping (3.5) and (3.6), and modifying (3.2) accordingly. The resulting reduced set of equations and inequalities will lead to relatively shorter statements and proofs for many of the sufficiency principles that will be developed in this study.

In the proofs of our sufficiency theorems, we shall make frequent use of two auxiliary results, namely, the generalized Cauchy inequality and an alternative expression for the objective function of (P). These results are formally stated in the following lemmas.

Lemma 4.1. [26] *For each $a, b \in \mathbb{R}^m$, $\langle a, b\rangle \leq \|a\|^* \|b\|$.*

Lemma 4.2. *For each $x \in X$,*

$$\varphi(x) \equiv \max_{y\in Y} \frac{f(x,y) + \|A(y)x\|_a}{g(x,y) - \|B(y)x\|_b} = \max_{p\in \underline{n+1}} \max_{\substack{u\in U \\ y^i \in Y}} \frac{\sum\limits_{i=1}^{p} u_i[f(x,y^i) + \|A(y^i)x\|_a]}{\sum\limits_{i=1}^{p} u_i[g(x,y^i) - \|B(y^i)x\|_b]}.$$

Proof. Since

$$\varphi(x) \equiv \max_{y\in Y} \frac{f(x,y) + \|A(y)x\|_a}{g(x,y) - \|B(y)x\|_b},$$

it is clear that for any $p \in \underline{n+1}, u \in U,\ y^i \in Y, i \in \underline{p}$, we have

$$\varphi(x) \geq \frac{f(x,y^i) + \|A(y^i)x\|_a}{g(x,y^i) - \|B(y^i)x\|_b},$$

and hence

$$\varphi(x) u_i[g(x,y^i) - \|B(y^i)x\|_b] \geq u_i[f(x,y^i) + \|A(y^i)x\|_a],$$

which implies

$$\varphi(x) \geq \frac{\sum\limits_{i=1}^{p} u_i[f(x,y^i) + \|A(y^i)x\|_a]}{\sum\limits_{i=1}^{p} u_i[g(x,y^i) - \|B(y^i)x\|_b]}.$$

From this inequality we conclude that

$$\varphi(x) \geq \max_{p\in\underline{n+1}} \max_{\substack{u\in U\\ y^i\in Y}} \frac{\sum\limits_{i=1}^{p} u_i[f(x,y^i)+\|A(y^i)x\|_a]}{\sum\limits_{i=1}^{p} u_i[g(x,y^i)-\|B(y^i)x\|_b]}.$$

On the other hand, since the set Y is compact, there is $y^*\in Y$, such that

$$\varphi(x) = \frac{f(x,y^*)+\|A(y^*)x\|_a}{g(x,y^*)-\|B(y^*)x\|_b},$$

and, therefore, it follows that

$$\varphi(x) \leq \max_{p\in\underline{n+1}} \max_{\substack{u\in U\\ y^i\in Y}} \frac{\sum\limits_{i=1}^{p} u_i[f(x,y^i)+\|A(y^i)x\|_a]}{\sum\limits_{i=1}^{p} u_i[g(x,y^i)-\|B(y^i)x\|_b]}.$$

Now the assertion of the lemma follows from the last two inequalities. □

To simplify the ensuing presentation, we use the following list of symbols:

$$\begin{aligned}
\mathcal{A}_i(x,\bar{y},\alpha) &= f(x,y^i)+\langle\alpha^i, A(y^i)x\rangle, \quad i\in\underline{p},\\
\mathcal{B}_i(x,\bar{y},\beta) &= -g(x,y^i)+\langle\beta^i, B(y^i)x\rangle, \quad i\in\underline{p},\\
\mathcal{C}_j(x,\gamma) &= G_j(x)+\langle\gamma^j, C_jx\rangle, \quad j\in\underline{q},\\
\mathcal{D}_k(x,w) &= w_kH_k(x), \quad k\in\underline{r},\\
\mathcal{C}(x,v,\gamma) &= \sum_{j=1}^{q} v_j[G_j(x)+\langle\gamma^j, C_jx\rangle],\\
\mathcal{D}(x,w) &= \sum_{k=1}^{r} w_kH_k(x),\\
\mathcal{G}(x,v,w,\gamma) &= \sum_{j=1}^{q} v_j[G_j(x)+\langle\gamma^j, C_jx\rangle]+\sum_{k=1}^{r} w_kH_k(x),\\
I_+(u) &= \{i\in\underline{p} : u_i>0\} \text{ for fixed } u\in U,\\
J_+(v) &= \{j\in\underline{q} : v_j>0\} \text{ for fixed } v\in\mathbb{R}^q_+,\\
K_*(w) &= \{k\in\underline{r} : w_k\neq 0\} \text{ for fixed } w\in\mathbb{R}^r,\\
\gamma &= (\gamma^1,\gamma^2,\ldots,\gamma^q).
\end{aligned}$$

We begin our discussion of sufficiency criteria for (P) with a collection of results in which separate $(\mathcal{F},b,\phi,\rho,\theta)$-univexity conditions are imposed on the functions $\mathcal{A}_i(\cdot,\bar{y},\alpha)$ and $\mathcal{B}_i(\cdot,\bar{y},\beta)$, $i\in\underline{p}$, whereas different types of generalized $(\mathcal{F},b,\phi,\rho,\theta)$-univexity assumptions are placed on certain combinations of the constraint functions.

Let

$$\mathbb{K} = \Big\{(p, \bar{y}, u, \alpha, \beta) : 1 \le p \le n+1;\ \bar{y} = (y^1, y^2, \ldots, y^p), y^i \in Y;\ u \in \mathbb{R}^p_+,$$
$$\sum_{i=1}^{p} u_i = 1;\ \alpha = (\alpha^1, \alpha^2, \ldots, \alpha^p), \alpha^i \in \mathbb{R}^\ell;\ \beta = (\beta^1, \beta^2, \ldots, \beta^p),\ \beta^i \in \mathbb{R}^m\Big\}.$$

Theorem 4.1. *Let $x^* \in \mathbb{F}$, let $\lambda^* = \varphi(x^*) \ge 0$, and assume that the functions $f(\cdot, y)$, $g(\cdot, y)$, $y \in Y$, G_j, $j \in \underline{q}$, and H_k, $k \in \underline{r}$, are differentiable at x^*, and that there exist $(p, \bar{y}^*, u^*, \alpha^*, \beta^*) \in \mathbb{K}$, $v^* \in \mathbb{R}^q_+$, $w^* \in \mathbb{R}^r$, and $\gamma^{*j} \in \mathbb{R}^{n_j}$, $j \in \underline{q}$, such that*

$$\sum_{i=1}^{p} u_i^* \{\nabla f(x^*, y^{*i}) + A^T(y^{*i})\alpha^{*i} - \lambda^*[\nabla g(x^*, y^{*i}) - B^T(y^{*i})\beta^{*i}]\}$$
$$+ \sum_{j=1}^{q} v_j^*[\nabla G_j(x^*) + C_j^T \gamma^{*j}] + \sum_{k=1}^{r} w_k^* \nabla H_k(x^*) = 0, \quad (4.1)$$

$$u_i^*\{f(x^*, y^{*i}) + \langle \alpha^{*i}, A(y^{*i})x^*\rangle - \lambda^*[g(x^*, y^{*i}) - \langle \beta^{*i}, B(y^{*i})x^*\rangle]\} = 0, \quad i \in \underline{p}, \quad (4.2)$$

$$v_j^*[G_j(x^*) + \langle \gamma^{*j}, C_j x^*\rangle] = 0, \quad j \in \underline{q}, \quad (4.3)$$

$$\|\alpha^{*i}\|_a^* \le 1, \quad \|\beta^{*i}\|_b^* \le 1, \quad i \in \underline{p}, \quad (4.4)$$

$$\|\gamma^{*j}\|_{c(j)}^* \le 1, \quad j \in \underline{q}. \quad (4.5)$$

Assume, furthermore, that any one of the following six sets of conditions holds:

(a) (i) *for each $i \in \underline{p}$, $\mathcal{A}_i(\cdot, \bar{y}^*, \alpha^*)$ is $(\mathcal{F}, b, \phi, \bar{\rho}_i, \theta)$-univex and $\mathcal{B}_i(\cdot, \bar{y}^*, \beta^*)$ is $(\mathcal{F}, b, \phi, \tilde{\rho}_i, \theta)$-univex at x^*, ϕ is superlinear, and $\phi(a) \ge 0 \Rightarrow a \ge 0$;*

(ii) *for each $j \in J_+ \equiv J_+(v^*)$, $\mathcal{C}_j(\cdot, \gamma^*)$ is $(\mathcal{F}, b, \hat{\phi}_j, \hat{\rho}_j, \theta)$-quasiunivex at x^*, $\hat{\phi}_j$ is increasing, and $\hat{\phi}_j(0) = 0$;*

(iii) *for each $k \in K_* \equiv K_*(w^*)$, $\mathcal{D}_k(\cdot, w^*)$ is $(\mathcal{F}, b, \breve{\phi}_k, \breve{\rho}_k, \theta)$-quasiunivex at x^* and $\breve{\phi}_k(0) = 0$;*

(iv) *$\rho^*(x, x^*) + \sum\limits_{j \in J_+} v_j^* \hat{\rho}_j(x, x^*) + \sum\limits_{k \in K_*} \breve{\rho}_k(x, x^*) \ge 0$ for all $x \in \mathbb{F}$, where $\rho^*(x, x^*) = \sum\limits_{i=1}^{p} u_i^*[\bar{\rho}_i(x, x^*) + \lambda^* \tilde{\rho}_i(x, x^*)]$;*

(b) (i) *for each $i \in \underline{p}$, $\mathcal{A}_i(\cdot, \bar{y}^*, \alpha^*)$ is $(\mathcal{F}, b, \phi, \bar{\rho}_i, \theta)$-univex and $\mathcal{B}_i(\cdot, \bar{y}^*, \beta^*)$ is $(\mathcal{F}, b, \phi, \tilde{\rho}_i, \theta)$-univex at x^*, ϕ is superlinear, and $\phi(a) \ge 0 \Rightarrow a \ge 0$;*

(ii) *$\mathcal{C}(\cdot, v^*, \gamma^*)$ is $(\mathcal{F}, b, \hat{\phi}, \hat{\rho}, \theta)$-quasiunivex at x^*, $\hat{\phi}$ is increasing, and $\hat{\phi}(0) = 0$;*

(iii) *for each $k \in K_*$, $\mathcal{D}_k(\cdot, w^*)$ is $(\mathcal{F}, b, \breve{\phi}_k, \breve{\rho}_k, \theta)$-quasiunivex at x^* and $\breve{\phi}_k(0) = 0$;*

(iv) *$\rho^*(x, x^*) + \hat{\rho}(x, x^*) + \sum\limits_{k \in K_*} \breve{\rho}_k(x, x^*) \ge 0$ for all $x \in \mathbb{F}$;*

(c) (i) *for each $i \in \underline{p}$, $\mathcal{A}_i(\cdot, \bar{y}^*, \alpha^*)$ is $(\mathcal{F}, b, \phi, \bar{\rho}_i, \theta)$-univex and $\mathcal{B}_i(\cdot, \bar{y}^*, \beta^*)$ is $(\mathcal{F}, b, \phi, \tilde{\rho}_i, \theta)$-univex at x^*, ϕ is superlinear, and $\phi(a) \ge 0 \Rightarrow a \ge 0$;*

(ii) *for each $j \in J_+$, $\mathcal{C}_j(\cdot, \gamma^*)$ is $(\mathcal{F}, b, \hat{\phi}_j, \hat{\rho}_j, \theta)$-quasiunivex at x^*, $\hat{\phi}_j$ is increasing, and $\hat{\phi}_j(0) = 0$;*

(iii) *$\mathcal{D}(\cdot, w^*)$ is $(\mathcal{F}, \beta, \breve{\phi}, \breve{\rho}, \theta)$-quasiunivex at x^* and $\breve{\phi}(0) = 0$;*

(iv) $\rho^*(x,x^*)+\sum\limits_{j\in J_+} v_j^*\hat{\rho}_j(x,x^*)+\breve{\rho}(x,x^*)\geq 0$ *for all* $x\in\mathbb{F}$;

(d) (i) *for each* $i\in\underline{p}$, $\mathcal{A}_i(\cdot,\bar{y}^*,\alpha^*)$ *is* $(\mathcal{F},b,\phi,\bar{\rho}_i,\theta)$*-univex and* $\mathcal{B}_i(\cdot,\bar{y}^*,\beta^*)$ *is* $(\mathcal{F},b,\phi,\tilde{\rho}_i,\theta)$*-univex at* x^*, ϕ *is superlinear, and* $\phi(a)\geq 0\Rightarrow a\geq 0$;

(ii) $\mathcal{C}(\cdot,v^*,\gamma^*)$ *is* $(\mathcal{F},b,\hat{\phi},\hat{\rho},\theta)$*-quasiunivex at* x^*, $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0)=0$;

(iii) $\mathcal{D}(\cdot,w^*)$ *is* $(\mathcal{F},b,\breve{\phi},\breve{\rho},\theta)$*-quasiunivex at* x^* *and* $\breve{\phi}(0)=0$;

(iv) $\rho^*(x,x^*)+\hat{\rho}(x,x^*)+\breve{\rho}(x,x^*)\geq 0$ *for all* $x\in\mathbb{F}$;

(e) (i) *for each* $i\in\underline{p}$, $\mathcal{A}_i(\cdot,\bar{y}^*,\alpha^*)$ *is* $(\mathcal{F},b,\phi,\bar{\rho}_i,\theta)$*-univex and* $\mathcal{B}_i(\cdot,\bar{y}^*,\beta^*)$ *is* $(\mathcal{F},b,\phi,\tilde{\rho}_i,\theta)$*-univex at* x^*, ϕ *is superlinear, and* $\phi(a)\geq 0\Rightarrow a\geq 0$;

(ii) $\mathcal{G}(\cdot,v^*,w^*,\gamma^*)$ *is* $(\mathcal{F},b,\hat{\phi},\hat{\rho},\theta)$*-quasiunivex at* x^*, $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0)=0$;

(iii) $\rho^*(x,x^*)+\hat{\rho}(x,x^*)\geq 0$ *for all* $x\in\mathbb{F}$;

(f) *the Lagrangian-type function*

$$z\to L(z,\bar{y}^*,\lambda^*,u^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*)=\sum_{i=1}^{p}u_i^*\{f(z,y^{*i})+\langle\alpha^{*i},A(y^{*i})z\rangle$$
$$-\lambda^*[g(z,y^{*i})-\langle\beta^{*i},B(y^{*i})z\rangle]\}+\sum_{j=1}^{q}v_j^*[G_j(z)+\langle\gamma^{*j},C_jz\rangle]+\sum_{k=1}^{r}w_k^*H_k(z)$$

is $(\mathcal{F},b,\phi,0,\theta)$*-pseudounivex at* x^* *and* $\phi(a)\geq 0\Rightarrow a\geq 0$.

Then x^* *is an optimal solution of* (P).

Proof. Let x be an arbitrary feasible solution of (P).
(a): Using the hypotheses specified in (i), we have for each $i\in\underline{p}$,

$$\phi\big(f(x,y^{*i})+\langle\alpha^{*i},A(y^{*i})x\rangle-[f(x^*,y^{*i})+\langle\alpha^{*i},A(y^{*i})x^*\rangle]\big)$$
$$\geq\mathcal{F}\big(x,x^*;b(x,x^*)[\nabla f(x^*,y^{*i})+A(y^{*i})^T\alpha^{*i}]\big)+\bar{\rho}_i(x,x^*)\|\theta(x,x^*)\|^2$$

and

$$\phi\big(-[g(x,y^{*i})-\langle\beta^{*i},B(y^{*i})x\rangle]+g(x^*,y^{*i})-\langle\beta^{*i},B(y^{*i})x^*\rangle\big)$$
$$\geq\mathcal{F}\big(x,x^*;-b(x,x^*)[\nabla g(x^*,y^{*i})-B(y^{*i})^T\beta^{*i}]\big)+\tilde{\rho}_i(x,x^*)\|\theta(x,x^*)\|^2.$$

Inasmuch as $\lambda^*\geq 0$, $u^*\geq 0$, $\sum\limits_{i=1}^{p}u_i^*=1$, ϕ is superlinear, and $\mathcal{F}(x,x^*;\cdot)$ is sublinear, we deduce from the above inequalities that

$$\phi\Big(\sum_{i=1}^{p}u_i^*\Big\{f(x,y^{*i})+\langle\alpha^{*i},A(y^{*i})x\rangle-\lambda^*[g(x,y^{*i})-\langle\beta^{*i},B(y^{*i})x\rangle]$$
$$-\{f(x^*,y^{*i})+\langle\alpha^{*i},A(y^{*i})x^*\rangle-\lambda^*[g(x^*,y^{*i})-\langle\beta^{*i},B(y^{*i})x^*\rangle]\}\Big\}\Big)$$
$$\geq\mathcal{F}\Big(x,x^*;b(x,x^*)\sum_{i=1}^{p}u_i^*\{\nabla f(x^*,y^{*i})+A(y^{*i})^T\alpha^{*i}$$
$$-\lambda^*[\nabla g(x^*,y^{*i})-B(y^{*i})^T\beta^{*i}]\}\Big)+\sum_{i=1}^{p}u_i^*[\bar{\rho}_i(x,x^*)+\lambda^*\tilde{\rho}_i(x,x^*)]\|\theta(x,x^*)\|^2. \quad (4.6)$$

Since for each $j \in J_+$,

$$\begin{aligned} G_j(x) + \langle \gamma^{*j}, C_j x\rangle &\le G_j(x) + \|\gamma^{*j}\|^*_{c(j)} \|C_j x\|_{c(j)} \quad \text{(by Lemma 4.1)} \\ &\le G_j(x) + \|C_j x\|_{c(j)} \quad \text{(by (4.5))} \\ &\le 0 \quad \text{(since } x \in \mathbb{F}\text{)} \\ &= G_j(x^*) + \langle \gamma^{*j}, C_j x^*\rangle \quad \text{(by (4.3))}, \end{aligned}$$

in view of the properties of the functions $\hat{\phi}_j,\ j \in \underline{q}$, we get

$$\hat{\phi}_j\big(G_j(x) + \langle \gamma^{*j}, C_j x\rangle - [G_j(x^*) + \langle \gamma^{*j}, C_j x^*\rangle]\big) \le 0,$$

which by (ii) implies that

$$\mathcal{F}(x, x^*; b(x, x^*)[\nabla G_j(x^*) + C_j^T \gamma^{*j}]) \le -\hat{\rho}_j(x, x^*)\|\theta(x, x^*)\|^2.$$

As $v_j^* \ge 0$ for each $j \in \underline{q}$, $v_j^* = 0$ for each $j \in \underline{q}\backslash J_+$ (complement of J_+ relative to $\underline{q}$), and $\mathcal{F}(x, x^*; \cdot)$ is sublinear, the above inequalities yield

$$\mathcal{F}\Big(x, x^*; b(x, x^*) \sum_{j=1}^{q} v_j^*[\nabla G_j(x^*) + C_j^T \gamma^{*j}]\Big) \le -\sum_{j \in J_+} v_j^* \hat{\rho}_j(x, x^*)\|\theta(x, x^*)\|^2. \tag{4.7}$$

In a similar manner, we can show that (iii) leads to the following inequality:

$$\mathcal{F}\Big(x, x^*; b(x, x^*) \sum_{k=1}^{r} w_k^* \nabla H_k(x^*)\Big) \le -\sum_{k \in K_*} \breve{\rho}_k(x, x^*)\|\theta(x, x^*)\|^2. \tag{4.8}$$

Combining (4.1), (4.6), (4.7), and (4.8), and using (4.2), (iv), and the sublinearity of $\mathcal{F}(x, x^*; \cdot)$, we obtain

$$\phi\Big(\sum_{i=1}^{p} u_i^*\{f(x, y^{*i}) + \langle \alpha^{*i}, A(y^{*i})x\rangle - \lambda^*[g(x, y^{*i}) - \langle \beta^{*i}, B(y^{*i})x\rangle]\}\Big) \ge 0.$$

But $\phi(a) \ge 0 \Rightarrow a \ge 0$ and hence we have

$$\begin{aligned} 0 &\le \sum_{i=1}^{p} u_i^*\{f(x, y^{*i}) + \langle \alpha^{*i}, A(y^{*i})x\rangle - \lambda^*[g(x, y^{*i}) - \langle \beta^{*i}, B(y^{*i})x\rangle]\} \\ &\le \sum_{i=1}^{p} u_i^*\{f(x, y^{*i}) + \|\alpha^{*i}\|^*_a \|A(y^{*i})x\|_a - \lambda^*[g(x, y^{*i}) \\ &\quad - \|\beta^{*i}\|^*_b \|B(y^{*i})x\|_b]\} \ \text{(by Lemma 4.1)} \\ &\le \sum_{i=1}^{p} u_i^*\{f(x, y^{*i}) + \|A(y^{*i})x\|_a - \lambda^*[g(x, y^{*i}) - \|B(y^{*i})x\|_b]\} \\ &\quad \text{(by (4.4))}. \end{aligned} \tag{4.9}$$

Now using (4.9) and Lemma 4.2, we see that

$$\varphi(x^*) = \lambda^* \le \frac{\sum\limits_{i=1}^{p} u_i^*[f(x,y^{*i}) + \|A(y^{*i})x\|_a]}{\sum\limits_{i=1}^{p} u_i^*[g(x,y^{*i}) - \|B(y^{*i})x\|_b]}$$

$$\le \max_{p\in\underline{n+1}} \max_{\substack{d\in U \\ y^i\in Y}} \frac{\sum\limits_{i=1}^{p} d_i[f(x,y^i) + \|A(y^i)x\|_a]}{\sum\limits_{i=1}^{p} d_i[g(x,y^i) - \|B(y^i)x\|_b]} = \varphi(x).$$

Since $x \in \mathbb{F}$ was arbitrary, we conclude from this inequality that x^* is an optimal solution of (P).

(b): As shown in part (a), for each $j \in J_+$, we have

$$G_j(x) + \langle \gamma^{*j}, C_j x\rangle \le G_j(x^*) + \langle \gamma^{*j}, C_j x^*\rangle,$$

and hence using the properties of the function $\hat{\phi}$, we get

$$\hat{\phi}\Big(\sum_{j=1}^{q} v_j^*[G_j(x) + \langle \gamma^{*j}, C_j x\rangle] - \sum_{j=1}^{q} v_j^*[G_j(x^*) + \langle \gamma^{*j}, C_j x^*\rangle]\Big) \le 0,$$

which in view of (ii) implies that

$$\mathcal{F}\Big(x, x^*; b(x,x^*)\sum_{j=1}^{q} v_j^*[\nabla G_j(x^*) + C_j^T\gamma^{*j}]\Big) \le -\hat{\rho}(x,x^*)\|\theta(x,x^*)\|^2.$$

Now proceeding as in the proof of part (a) and using this inequality instead of (4.7), we arrive at (4.9), which leads to the desired conclusion that x^* is an optimal solution of (P).

(c) - (e): The proofs are similar to those of parts (a) and (b).

(f): By our $(\mathcal{F}, b, \phi, 0, \theta)$-pseudounivexity assumption and the sublinearity of $\mathcal{F}(x, x^*; \cdot)$, (4.1) implies that

$$\phi\big(L(x,\bar{y}^*,\lambda^*,u^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*) - L(x^*,\bar{y}^*,\lambda^*,u^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*)\big) \ge 0.$$

But $\phi(a) \ge 0 \Rightarrow a \ge 0$ and hence we have

$$L(x,\bar{y}^*,\lambda^*,u^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*) \ge L(x^*,\bar{y}^*,\lambda^*,u^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*,\lambda^*).$$

Because $x, x^* \in \mathbb{F}$, $v^* \ge 0$, and (4.2) and (4.3) hold, the right-hand side of the above inequality is equal to zero, and so we get

$$0 \le \sum_{i=1}^{p} u_i^*\{f(x,y^{*i}) + \langle \alpha^{*i}, A(y^{*i})x\rangle - \lambda^*[g(x,y^{*i}) - \langle \beta^{*i}, B(y^{*i})x\rangle]\}$$
$$+ \sum_{j=1}^{q} v_j^*[G_j(x) + \langle \gamma^{*j}, C_j x\rangle]$$
$$\le \sum_{i=1}^{p} u_i^*\{f(x,y^{*i}) + \|\alpha^{*i}\|_a^*\|A(y^{*i})x\|_a - \lambda^*[g(x,y^{*i})$$

$$- \|\beta^{*i}\|_b^* \|B(y^{*i})x\|_b]\} + \sum_{j=1}^{q} v_j^*[G_j(x) + \|\gamma^{*j}\|_{c(j)}^* \|C_j x\|_{c(j)}]$$

(by Lemma 4.1)

$$\leq \sum_{i=1}^{p} u_i^*\{f(x, y^{*i}) + \|A(y^{*i})x\|_a - \lambda^*[g(x, y^{*i}) - \|B(y^{*i})x\|_b]\}$$

$$+ \sum_{j=1}^{q} v_j^*[G_j(x) + \|C_j x\|_{c(j)}] \quad \text{(by (4.4) and (4.5))}$$

$$\leq \sum_{i=1}^{p} u_i^*\{f(x, y^{*i}) + \|A(y^{*i})x\|_a - \lambda^*[g_i(x) - \|B_i x\|_b]\}$$

(by the feasibility of x),

which is precisely (4.9). As seen in the proof of part (a), this inequality leads to the desired conclusion that x^* is an optimal solution of (P). □

The following variant of Theorem 4.1 can be proved in an identical manner.

Theorem 4.2. *Let $x^* \in \mathbb{F}$, let $\lambda^* = \varphi(x^*) \geq 0$, and assume that the functions $f(\cdot, y)$, $g(\cdot, y)$, $y \in Y$, G_j, $j \in \underline{q}$, and H_k, $k \in \underline{r}$, are differentiable at x^*, and that there exist $(p, \bar{y}^*, u^*, \alpha^*, \beta^*) \in \mathbb{K}$, $v^* \in \mathbb{R}_+^q$, $w^* \in \mathbb{R}^r$, and $\gamma^{*j} \in \mathbb{R}^{n_j}$, $j \in \underline{q}$, such that (4.2) - (4.5) and the following inequality hold:*

$$\mathcal{F}\Big(x, x^*; \sum_{i=1}^{p} u_i^*\{\nabla f_i(x^*, y^{*i}) + A(y^{*i})^T \alpha^{*i} - \lambda^*[\nabla g_i(x^*, y^{*i}) - B(y^{*i})^T \beta^{*i}]\}$$

$$+ \sum_{j=1}^{q} v_j^*[\nabla G_j(x^*) + C_j^T \gamma^{*j}] + \sum_{k=1}^{r} w_k^* \nabla H_k(x^*)\Big) \geq 0 \quad \textit{for all } x \in \mathbb{F}, \quad (4.10)$$

where $\mathcal{F}(x, x^*; \cdot)\colon \mathbb{R}^n \to \mathbb{R}$ is a given sublinear function. Furthermore, assume that any one of the six sets of conditions specified in Theorem 4.1 *is satisfied. Then x^* is an optimal solution of (P).*

Although the proofs of Theorems 4.1 and 4.2 are essentially the same, their contents are somewhat different. This can easily be seen by comparing (4.1) with (4.10). We observe that any octuple $(x^*, \lambda^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ that satisfies (4.1) - (4.5) also satisfies (4.2) - (4.5), and (4.10), but the converse is not necessarily true. Moreover, (4.1) is a system of n equations, whereas (4.10) is a single inequality. Evidently, from a computational point of view, (4.1) is preferable to (4.10) because of the dependence of the latter on the feasible set of (P).

In Theorem 4.1, separate $(\mathcal{F}, b, \phi, \rho, \theta)$-univexity assumptions were imposed on the functions $\mathcal{A}_i(\cdot, \bar{y}^*, \alpha^*)$ and $\mathcal{B}_i(\cdot, \bar{y}^*, \beta^*)$, $i \in \underline{p}$. It is possible to formulate a great variety of additional sets of sufficient optimality conditions in which various generalized $(\mathcal{F}, b, \phi, \rho, \theta)$-univexity requirements are placed on certain combinations of these and the constraint functions. However, it turns out that a great majority of these results are subsumed in the generalized sufficiency criteria that will be established in the next section and, therefore, they will not be discussed separately.

5. GENERALIZED SUFFICIENCY CRITERIA

In this section, we discuss several families of sufficient optimality results under various generalized $(\mathcal{F}, b, \phi, \rho, \theta)$-univexity hypotheses imposed on certain combinations of the problem functions. This is accomplished by employing a certain partitioning scheme which was originally proposed in [41] for the purpose of constructing generalized dual problems for nonlinear programming problems. For this we need some additional notation.

Let $\{J_0, J_1, \ldots, J_m\}$ and $\{K_0, K_1, \ldots, K_m\}$ be partitions of the index sets $\underline{q}$ and $\underline{r}$, respectively; thus, $J_\mu \subseteq \underline{q}$ for each $\mu \in \underline{m} \cup \{0\}$, $J_\mu \cap J_\nu = \emptyset$ for each $\mu, \nu \in \underline{m} \cup \{0\}$ with $\mu \neq \nu$, and $\bigcup_{\mu=0}^{m} J_\mu = \underline{q}$. Obviously, similar properties hold for $\{K_0, K_1, \ldots, K_m\}$. Moreover, if m_1 and m_2 are the numbers of the partitioning sets of $\underline{q}$ and $\underline{r}$, respectively, then $m = \max\{m_1, m_2\}$ and $J_\mu = \emptyset$ or $K_\mu = \emptyset$ for $\mu > \min\{m_1, m_2\}$.

In addition, we use the real-valued functions $\Phi_i(\cdot, \bar{y}, \lambda, v, w, \alpha, \beta, \gamma)$, $i \in \underline{p}$, $\Phi(\cdot, \bar{y}, \lambda, u, v, w, \alpha, \beta, \gamma)$, and $\Lambda_t(\cdot, v, w, \gamma)$ defined, for fixed $\bar{y}, \lambda, u, v, w, \alpha, \beta$, and γ, on X as follows:

$$\Phi_i(x, \bar{y}, \lambda, v, w, \alpha, \beta, \gamma) = f(x, y^i) + \langle \alpha^i, A(y^i)x\rangle - \lambda[g(x, y^i) - \langle \beta^i, B(y^i)x\rangle] + \sum_{j \in J_0} v_j[G_j(x) + \langle \gamma^j, C_j x\rangle] + \sum_{k \in K_0} w_k H_k(x), \quad i \in \underline{p},$$

$$\Phi(x, \bar{y}, \lambda, u, v, w, \alpha, \beta, \gamma) = \sum_{i=1}^{p} u_i\{f(x, y^i) + \langle \alpha^i, A(y^i)x\rangle - \lambda[g(x, y^i) - \langle \beta^i, B(y^i)x\rangle]\} + \sum_{j \in J_0} v_j[G_j(x) + \langle \gamma^j, C_j x\rangle] + \sum_{k \in K_0} w_k H_k(x),$$

$$\Lambda_t(x, v, w, \gamma) = \sum_{j \in J_t} v_j[G_j(x) + \langle \gamma^j, C_j x\rangle] + \sum_{k \in K_t} w_k H_k(x), \quad t \in \underline{m}.$$

Making use of the sets and functions defined above, we can now formulate our first collection of generalized sufficiency results for (P) as follows.

Theorem 5.1. *Let $x^* \in \mathbb{F}$, let $\lambda^* = \varphi(x^*) \geq 0$, and assume that the functions $f(\cdot, y)$, $g(\cdot, y)$, $y \in Y$, G_j, $j \in \underline{q}$, and H_k, $k \in \underline{r}$, are differentiable at x^*, and that there exist $(p, \bar{y}^*, u^*, \alpha^*, \beta^*) \in \mathbb{K}$, $v^* \in \mathbb{R}^q_+$, $w^* \in \mathbb{R}^r$, and $\gamma^{*j} \in \mathbb{R}^{n_j}$, $j \in \underline{q}$, such that (4.1) - (4.5) hold. Assume, furthermore, that any one of the following four sets of hypotheses is satisfied:*

(a) (i) *$\Phi(\cdot, \bar{y}^*, \lambda^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ is prestrictly $(\mathcal{F}, b, \bar{\phi}, \bar{\rho}, \theta)$-quasiunivex at x^* and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;*

(ii) *for each $t \in \underline{m}$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ is strictly $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$-pseudounivex at x^*, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$;*

(iii) *$\bar{\rho}(x, x^*) + \sum_{t=1}^{m} \tilde{\rho}_t(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;*

(b) (i) *$\Phi(\cdot, \bar{y}^*, \lambda^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ is $(\mathcal{F}, b, \bar{\phi}, \bar{\rho}, \theta)$-pseudounivex at x^* and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;*

(ii) *for each $t \in \underline{m}$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ is $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$-quasiunivex at x^*, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$;*

(iii) *$\bar{\rho}(x, x^*) + \sum_{t=1}^{m} \tilde{\rho}_t(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;*

(c) (i) $\Phi(\cdot, \bar{y}^*, \lambda^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ *is prestrictly* $(\mathcal{F}, b, \bar{\phi}, \bar{\rho}, \theta)$*-quasiunivex at* x^* *and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$*;*

(ii) *for each* $t \in \underline{m}$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ *is* $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$*-quasiunivex at* x^*, $\tilde{\phi}_t$ *is increasing, and* $\tilde{\phi}_t(0) = 0$*;*

(iii) $\bar{\rho}(x, x^*) + \sum\limits_{t=1}^{m} \tilde{\rho}_t(x, x^*) > 0$ *for all* $x \in \mathbb{F}$*;*

(d) (i) $\Phi(\cdot, \bar{y}^*, \lambda^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ *is prestrictly* $(\mathcal{F}, b, \bar{\phi}, \bar{\rho}, \theta)$*-quasiunivex at* x^* *and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$*;*

(ii) *for each* $t \in \underline{m_1}$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ *is* $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$*-quasiunivex at* x^**, for each* $t \in \underline{m_2} \neq \emptyset$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ *is strictly* $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$*-pseudounivex at* x^**, and for each* $t \in \underline{m}$, $\tilde{\phi}_t$ *is increasing and* $\tilde{\phi}_t(0) = 0$*, where* $\{\underline{m_1}, \underline{m_2}\}$ *is a partition of* $\underline{m}$*;*

(iii) $\bar{\rho}(x, x^*) + \sum\limits_{t=1}^{m} \tilde{\rho}_t(x, x^*) \geq 0$ *for all* $x \in \mathbb{F}$.

Then x^* *is an optimal solution of* (P).

Proof. Let x be an arbitrary feasible solution of (P).

(a): Since $\mathcal{F}(x, x^*; \cdot)$ is sublinear and $b(x, x^*) > 0$, from (4.1) it follows that

$$\begin{aligned}
&\mathcal{F}\Big(x, x^*; b(x, x^*)\Big\{ \sum_{i=1}^{p} u_i^* \{\nabla f(x^*, y^{*i}) + A(y^{*i})^T \alpha^{*i} - \lambda^* [\nabla g(x^*, y^{*i}) \\
&\qquad - B(y^{*i})^T \beta^{*i}]\} + \sum_{j \in J_0} v_j^* [\nabla G_j(x^*) + C_j^T \gamma^{*j}] + \sum_{k \in K_0} w_k^* \nabla H_k(x^*) \Big\}\Big) \\
&\quad + \mathcal{F}\Big(x, x^*; b(x, x^*) \sum_{t=1}^{m} \Big\{ \sum_{j \in J_t} v_j^* [\nabla G_j(x^*) + C_j^T \gamma^{*j}] + \sum_{k \in K_t} w_k^* \nabla H_k(x^*) \Big\}\Big) \geq 0. \quad (5.1)
\end{aligned}$$

Since for each $t \in \underline{m}$,

$$\begin{aligned}
&\Lambda_t(x, v^*, w^*, \gamma^*) \\
&= \sum_{j \in J_t} v_j^* [G_j(x) + \langle \gamma^{*j}, C_j x \rangle] + \sum_{k \in K_t} w_k^* H_k(x) \\
&\leq \sum_{j \in J_t} v_j^* [G_j(x) + \|\gamma^{*j}\|_{c(j)}^* \|C_j x\|_{c(j)}] + \sum_{k \in K_t} w_k^* H_k(x) \\
&\quad \text{(by Lemma 4.1 and the nonnegativity of } v^*) \\
&\leq \sum_{j \in J_t} v_j^* [G_j(x) + \|C_j x\|_{c(j)}] + \sum_{k \in K_t} w_k^* H_k(x) \ \text{ (by (4.5))} \\
&\leq 0 \quad \text{(by the feasibility of } x) \\
&= \sum_{j \in J_t} v_j^* [G_j(x^*) + \langle \gamma^{*j}, C_j x^* \rangle] + \sum_{k \in K_t} w_k^* H_k(x^*) \\
&\quad \text{(by (4.3) and the feasibility of } x^*) \\
&= \Lambda_t(x^*, v^*, w^*, \gamma^*),
\end{aligned}$$

and hence $\tilde{\phi}_t\big(\Lambda_t(x,v^*,w^*,\gamma^*)-\Lambda_t(x^*,v^*,w^*,\gamma^*)\big)\le 0$, it follows from (ii) that

$$\mathcal{F}\Big(x,x^*;b(x,x^*)\Big\{\sum_{j\in J_t}v_j^*[\nabla G_j(x^*)+C_j^T\gamma^{*j}]+\sum_{k\in K_t}w_k^*\nabla H_k(x^*)\Big\}\Big)<-\tilde{\rho}_t(x,x^*)\|\theta(x,x^*)\|^2.$$

Summing over t and using the sublinearity of $\mathcal{F}(x,x^*;\cdot)$, we obtain

$$\mathcal{F}\Big(x,x^*;b(x,x^*)\sum_{t=1}^{m}\Big\{\sum_{j\in J_t}v_j^*[\nabla G_j(x^*)+C_j^T\gamma^{*j}]+\sum_{k\in K_t}w_k^*\nabla H_k(x^*)\Big\}\Big)<-\sum_{t=1}^{m}\tilde{\rho}_t(x,x^*)\|\theta(x,x^*)\|^2. \quad (5.2)$$

Combining (5.1) and (5.2), and using (iii) we get

$$\begin{aligned}\mathcal{F}\Big(x,x^*;b(x,x^*)\Big\{\sum_{i=1}^{p}u_i^*\{\nabla f(x^*,y^{*i})+A(y^{*i})^T\alpha^{*i}-\lambda^*[\nabla g(x^*,y^{*i})\\ -B(y^{*i})^T\beta^{*i}]\}+\sum_{j\in J_0}v_j^*[\nabla G_j(x^*)+C_j^T\gamma^{*j}]+\sum_{k\in K_0}w_k^*\nabla H_k(x^*)\Big\}\Big)\\ >\sum_{t=1}^{m}\tilde{\rho}_t(x,x^*)\|\theta(x,x^*)\|^2\ge-\bar{\rho}(x,x^*)\|\theta(x,x^*)\|^2, \quad (5.3)\end{aligned}$$

which by virtue of (i) implies that

$$\bar{\phi}\big(\Phi(x,\bar{y}^*,\lambda^*,u^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*)-\Phi(x^*,\bar{y}^*,\lambda^*,u^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*)\big)\ge 0.$$

But $\bar{\phi}(a)\ge 0\Rightarrow a\ge 0$, and hence we get

$$\Phi(x,\bar{y}^*,\lambda^*,u^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*)\ge\Phi(x^*,\bar{y}^*,\lambda^*,u^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*)=0,$$

where the equality follows from (4.2), (4.3), and the feasibility of x^*. Therefore, we have

$$\begin{aligned}0&\le\Phi(x,\bar{y}^*,\lambda^*,u^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*)\\ &=\sum_{i=1}^{p}u_i^*\{f(x,y^{*i})+\langle\alpha^{*i},A(y^{*i})x\rangle-\lambda^*[g(x,y^{*i})-\langle\beta^{*i},B(y^{*i})x\rangle]\}\\ &\quad+\sum_{j\in J_0}v_j^*[G_j(x)+\langle\gamma^{*j},C_jx\rangle]+\sum_{k\in K_0}w_k^*H_k(x)\\ &\le\sum_{i=1}^{p}u_i^*\{f(x,y^{*i})+\|\alpha^{*i}\|_a^*\|A(y^{*i})x\|_a-\lambda^*[g(x,y^{*i})\\ &\quad-\|\beta^{*i}\|_b^*\|B(y^{*i})x\|_b]\}+\sum_{j\in J_0}v_j^*[G_j(x)+\|\gamma^{*j}\|_{c(j)}^*\|C_jx\|_{c(j)}]\\ &\quad\text{(by Lemma 4.1 and the feasibility of } x)\\ &\le\sum_{i=1}^{p}u_i^*\{f(x,y^{*i})+\|A(y^{*i})x\|_a-\lambda^*[g(x,y^{*i})-\|B(y^{*i})x\|_b]\}\\ &\quad+\sum_{j\in J_0}v_j^*[G_j(x)+\|C_jx\|_{c(j)}]\ \text{ (by (4.4) and (4.5))}\end{aligned}$$

$$\leq \sum_{i=1}^{p} u_i^* \{f(x, y^{*i}) + \|A(y^{*i})x\|_a - \lambda^*[g(x, y^{*i}) - \|B(y^{*i})x\|_b]\}$$

(by the feasibility of x).

Now using this inequality and Lemma 4.2, as in the proof of Theorem 4.1, we obtain $\varphi(x^*) \leq \varphi(x)$. Since x was arbitrary, we conclude that x^* is an optimal solution of (P).

(b): Proceeding as in the proof of part (a), we see that (ii) leads to the following inequality:

$$\mathcal{F}\Big(x, x^*; b(x, x^*) \sum_{t=1}^{m} \Big\{ \sum_{j \in J_t} v_j^* [\nabla G_j(x^*) + C_j^T \gamma^{*j}] + \sum_{k \in K_t} w_k^* \nabla H_k(x^*) \Big\}\Big) \leq -\sum_{t=1}^{m} \tilde{\rho}_t(x, x^*) \|\theta(x, x^*)\|^2.$$

Now combining this inequality with (5.1) and using (iii), we obtain

$$\mathcal{F}\Big(x, x^*; b(x, x^*) \Big\{ \sum_{i=1}^{p} u_i^* \{\nabla f(x^*, y^{*i}) + A(y^{*i})^T \alpha^{*i} - \lambda^* [\nabla g(x^*, y^{*i}) - B(y^{*i})^T \beta^{*i}]\} + \sum_{j \in J_0} v_j^* [\nabla G_j(x^*) + C_j^T \gamma^{*j}] + \sum_{k \in K_0} w_k^* \nabla H_k(x^*) \Big\}\Big) \geq \sum_{t=1}^{m} \tilde{\rho}_t(x, x^*) \|\theta(x, x^*)\|^2 \geq -\bar{\rho}(x, x^*) \|\theta(x, x^*)\|^2,$$

which by virtue of (i) implies that

$$\bar{\phi}\big(\Phi(x, \bar{y}^*, \lambda^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*) - \Phi(x^*, \bar{y}^*, \lambda^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)\big) \geq 0.$$

The rest of the proof is identical to that of part (a).

(c) - (d): The proofs are similar to those of parts (a) and (b). □

Theorem 5.2. *Let $x^* \in \mathbb{F}$, let $\lambda^* = \varphi(x^*) \geq 0$, and assume that the functions $f(\cdot, y)$, $g(\cdot, y)$, $y \in Y$, G_j, $j \in \underline{q}$, and H_k, $k \in \underline{r}$, are differentiable at x^*, and that there exist $(p, \bar{y}^*, u^*, \alpha^*, \beta^*) \in \mathbb{K}$, $v^* \in \mathbb{R}_+^q$, $w^* \in \mathbb{R}^r$, and $\gamma^{*j} \in \mathbb{R}^{n_j}$, $j \in \underline{q}$, such that (4.1) - (4.5) hold. Assume, furthermore, that any one of the following seven sets of hypotheses is satisfied:*

(a) (i) *for each $i \in I_+ \equiv I_+(u^*)$, $\Phi_i(\cdot, \bar{y}^*, \lambda^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ is $(\mathcal{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-pseudounivex at x^*, $\bar{\phi}_i$ is strictly increasing, and $\bar{\phi}_i(0) = 0$;*

(ii) *for each $t \in \underline{m}$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ is $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$-quasiunivex at x^*, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$;*

(iii) $\sum\limits_{i \in I_+} u_i^* \bar{\rho}_i(x, x^*) + \sum\limits_{t=1}^{m} \tilde{\rho}_t(x, x^*) \geq 0$ *for all $x \in \mathbb{F}$;*

(b) (i) *for each $i \in I_+$, $\Phi_i(\cdot, \bar{y}^*, \lambda^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ is prestrictly $(\mathcal{F}, b, \bar{\phi}_i, \bar{\rho}_i, \theta)$-quasiunivex at x^*, $\bar{\phi}_i$ is strictly increasing, and $\bar{\phi}_i(0) = 0$;*

(ii) *for each $t \in \underline{m}$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ is strictly $(\mathcal{F}, b, \tilde{\phi}_t, \tilde{\rho}_t, \theta)$-pseudounivex at x^*, $\tilde{\phi}_t$ is increasing, and $\tilde{\phi}_t(0) = 0$;*

(iii) $\sum_{i\in I_+} u_i^*\bar{\rho}_i(x,x^*) + \sum_{t=1}^{m}\tilde{\rho}_t(x,x^*) \geq 0$ *for all* $x \in \mathbb{F}$;

(c) (i) *for each* $i \in I_+$, $\Phi_i(\cdot,\bar{y}^*,\lambda^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*)$ *is prestrictly* $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-*quasiunivex at* x^*, $\bar{\phi}_i$ *is strictly increasing, and* $\bar{\phi}_i(0)=0$;

(ii) *for each* $t \in \underline{m}$, $\Lambda_t(\cdot,v^*,w^*,\gamma^*)$ *is* $(\mathcal{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$-*quasiunivex at* x^*, $\tilde{\phi}_t$ *is increasing, and* $\tilde{\phi}_t(0)=0$;

(iii) $\sum_{i\in I_+} u_i^*\bar{\rho}_i(x,x^*) + \sum_{t=1}^{m}\tilde{\rho}_t(x,x^*) > 0$ *for all* $x \in \mathbb{F}$;

(d) (i) *for each* $i \in I_{1+}$, $\Phi_i(\cdot,\bar{y}^*,\lambda^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*)$ *is* $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-*pseudounivex at* x^*, *for each* $i \in I_{2+}$, $\Phi_i(\cdot,\bar{y}^*,\lambda^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*)$ *is prestrictly* $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-*quasiunivex at* x^*, *and for each* $i \in I_+$, $\bar{\phi}_i$ *is strictly increasing and* $\bar{\phi}_i(0)=0$, *where* $\{I_{1+},I_{2+}\}$ *is a partition of* I_+;

(ii) *for each* $t \in \underline{m}$, $\Lambda_t(\cdot,v^*,w^*,\gamma^*)$ *is strictly* $(\mathcal{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$-*pseudounivex at* x^*, $\tilde{\phi}_t$ *is increasing, and* $\tilde{\phi}_t(0)=0$;

(iii) $\sum_{i\in I_+} u_i^*\bar{\rho}_i(x,x^*) + \sum_{t=1}^{m}\tilde{\rho}_t(x,x^*) \geq 0$ *for all* $x \in \mathbb{F}$;

(e) (i) *for each* $i \in I_{1+} \neq \emptyset$, $\Phi_i(\cdot,\bar{y}^*,\lambda^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*)$ *is* $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-*pseudounivex at* x^*, *for each* $i \in I_{2+}$, $\Phi_i(\cdot,\bar{y}^*,\lambda^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*)$ *is prestrictly* $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-*quasiunivex at* x^*, *and for each* $i \in I_+$, $\bar{\phi}_i$ *is strictly increasing and* $\bar{\phi}_i(0)=0$, *where* $\{I_{1+},I_{2+}\}$ *is a partition of* I_+;

(ii) *for each* $t \in \underline{m}$, $\Lambda_t(\cdot,v^*,w^*,\gamma^*)$ *is* $(\mathcal{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$-*quasiunivex at* x^*, $\tilde{\phi}_t$ *is increasing, and* $\tilde{\phi}_t(0)=0$;

(iii) $\sum_{i\in I_+} u_i^*\bar{\rho}_i(x,x^*) + \sum_{t=1}^{m}\tilde{\rho}_t(x,x^*) \geq 0$ *for all* $x \in \mathbb{F}$;

(f) (i) *for each* $i \in I_+$, $\Phi_i(\cdot,\bar{y}^*,\lambda^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*)$ *is prestrictly* $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-*quasiunivex at* x^*, $\bar{\phi}_i$ *is strictly increasing, and* $\bar{\phi}_i(0)=0$;

(ii) *for each* $t \in \underline{m}_1 \neq \emptyset$, $\Lambda_t(\cdot,v^*,w^*,\gamma^*)$ *is strictly* $(\mathcal{F},b,\bar{\phi}_t,\tilde{\rho}_t,\theta)$-*pseudounivex at* x^*, *for each* $t \in \underline{m}_2$, $\Lambda_t(\cdot,v^*,w^*,\gamma^*)$ *is* $(\mathcal{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$-*quasiunivex at* x^*, *and for each* $t \in \underline{m}$, $\tilde{\phi}_t$ *is increasing and* $\tilde{\phi}_t(0)=0$, *where* $\{\underline{m}_1,\underline{m}_2\}$ *is a partition of* $\underline{m}$;

(iii) $\sum_{i\in I_+} u_i^*\bar{\rho}_i(x,x^*) + \sum_{t=1}^{m}\tilde{\rho}_t(x,x^*) \geq 0$ *for all* $x \in \mathbb{F}$;

(g) (i) *for each* $i \in I_{1+}$, $\Phi_i(\cdot,\bar{y}^*,\lambda^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*)$ *is* $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-*pseudounivex at* x^*, *for each* $i \in I_{2+}$, $\Phi_i(\cdot,\bar{y}^*,\lambda^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*)$ *is prestrictly* $(\mathcal{F},b,\bar{\phi}_i,\bar{\rho}_i,\theta)$-*quasiunivex at* x^*, *and for each* $i \in I_+$, $\bar{\phi}_i$ *is strictly increasing and* $\bar{\phi}_i(0)=0$, *where* $\{I_{1+},I_{2+}\}$ *is a partition of* I_+;

(ii) *for each* $t \in \underline{m}_1$, $\Lambda_t(\cdot,v^*,w^*,\gamma^*)$ *is strictly* $(\mathcal{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$-*pseudounivex at* x^*, *for each* $t \in \underline{m}_2$, $\Lambda_t(\cdot,v^*,w^*,\gamma^*)$ *is* $(\mathcal{F},b,\tilde{\phi}_t,\tilde{\rho}_t,\theta)$-*quasiunivex at* x^*, *and for each* $t \in \underline{m}$, $\tilde{\phi}_t$ *is increasing and* $\tilde{\phi}_t(0)=0$, *where* $\{\underline{m}_1,\underline{m}_2\}$ *is a partition of* $\underline{m}$;

(iii) $\sum\limits_{i\in I_+} u_i^*\bar{\rho}_i(x,x^*)+\sum\limits_{t=1}^{m}\tilde{\rho}_t(x,x^*)\geq 0$ *for all* $x\in\mathbb{F}$;

(iv) $I_{1+}\neq\emptyset$, $\underline{m_1}\neq\emptyset$, *or* $\sum\limits_{i\in I_+} u_i^*\bar{\rho}_i(x,x^*)+\sum\limits_{t=1}^{m}\tilde{\rho}_t(x,x^*)>0$ *for all* $x\in\mathbb{F}$.

Then x^* *is an optimal solution of* (P).

Proof. (a): Suppose to the contrary that x^* is not an optimal solution of (P). Then there is a feasible solution $\bar{x}$ of (P) such that $\varphi(\bar{x})<\varphi(x^*)=\lambda^*$, which implies that

$$\frac{f(\bar{x},y^{*i})+\|A(y^{*i})\bar{x}\|_a}{g(\bar{x},y^{*i})-\|B(y^{*i})\bar{x}\|_b}<\lambda^*\quad\text{for each } i\in\underline{p}.$$

Hence, for each $i\in\underline{p}$, we have

$$f(\bar{x},y^{*i})+\|A(y^{*i})\bar{x}\|_a-\lambda^*[g(\bar{x},y^{*i})-\|B(y^{*i})\bar{x}\|_b]<0. \tag{5.4}$$

Keeping in mind that λ^* and v^* are nonnegative, we see that for each $i\in I_+$,

$$\begin{aligned}
&\Phi_i(\bar{x},\bar{y}^*,\lambda^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*)\\
&=f(\bar{x},y^{*i})+\langle\alpha^{*i},A(y^{*i})\bar{x}\rangle-\lambda^*[g(\bar{x},y^{*i})-\langle\beta^{*i},B(y^{*i})\bar{x}\rangle]\\
&\quad+\sum_{j\in J_0}v_j^*[G_j(\bar{x})+\langle\gamma^{*j},C_j\bar{x}\rangle]+\sum_{k\in K_0}w_k^*H_k(\bar{x})\\
&\leq f(\bar{x},y^{*i})+\|\alpha^{*i}\|_a^*\|A(y^{*i})\bar{x}\|_a-\lambda^*[g(\bar{x},y^{*i})-\|\beta^{*i}\|_b^*\|B(y^{*i})\bar{x}\|_b]\\
&\quad+\sum_{j\in J_0}v_j^*[G_j(\bar{x})+\|\gamma^{*j}\|_{c(j)}^*\|C_j\bar{x}\|_{c(j)}]\ \text{(by Lemma 4.1 and the feasibility of } \bar{x})\\
&\leq f(\bar{x},y^{*i})+\|A(y^{*i})\bar{x}\|_a-\lambda^*[g(\bar{x},y^{*i})-\|B(y^{*i})\bar{x}\|_b]\\
&\quad+\sum_{j\in J_0}v_j^*[G_j(\bar{x})+\|C_j\bar{x}\|_{c(j)}]\ \text{(by (4.4) and (4.5))}\\
&\leq f(\bar{x},y^{*i})+\|A(y^{*i})\bar{x}\|_a-\lambda^*[g(\bar{x},y^{*i})-\|B(y^{*i})\bar{x}\|_b]\\
&\quad\text{(by the feasibility of } \bar{x})\\
&<0\quad\text{(by (5.4))}\\
&=f(x^*,y^{*i})+\langle\alpha^{*i},A(y^{*i})x^*\rangle-\lambda^*[g(x^*,y^{*i})-\langle\beta^{*i},B(y^{*i})x^*\rangle]+\sum_{j\in J_0}v_j^*[G_j(x^*)\\
&\quad+\langle\gamma^{*j},C_jx^*\rangle]+\sum_{k\in K_0}w_k^*H_k(x^*)\ \text{(by (4.2), (4.3), and the feasibility of } x^*)\\
&=\Phi_i(x^*,\bar{y}^*,\lambda^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*),
\end{aligned}$$

and so using the properties of the function $\bar{\phi}_i$, we get

$$\bar{\phi}_i\big(\Phi_i(\bar{x},\bar{y}^*,\lambda^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*)-\Phi_i(x^*,\bar{y}^*,\lambda^*,v^*,w^*,\alpha^*,\beta^*,\gamma^*)\big)<0,$$

which in view of (i) implies that for each $i\in I_+$,

$$\begin{aligned}
\mathcal{F}\Big(\bar{x},x^*;b(\bar{x},x^*)\Big\{&\nabla f(x^*,y^{*i})+A(y^{*i})^T\alpha^{*i}-\lambda^*[\nabla g(x^*,y^{*i})-B(y^{*i})^T\beta^{*i}]\\
&+\sum_{j\in J_0}v_j^*[\nabla G_j(x^*)+C_j^T\gamma^{*j}]+\sum_{k\in K_0}w_k^*\nabla H_k(x^*)\Big\}\Big)<-\bar{\rho}_i(\bar{x},x^*)\|\theta(\bar{x},x^*)\|^2.
\end{aligned}$$

Since $u^* \geq 0,\ u_i^* = 0$ for each $i \in \underline{p}\backslash I_+,\ \sum\limits_{i=1}^{p} u_i^* = 1$, and $\mathcal{F}(\bar{x}, x^*; \cdot)$ is sublinear, the above inequalities yield

$$\mathcal{F}\Big(\bar{x}, x^*; b(\bar{x}, x^*)\Big\{ \sum_{i=1}^{p} u_i^* \{\nabla f(x^*, y^{*i}) + A(y^{*i})^T \alpha^{*i} - \lambda^* [\nabla g(x^*, y^{*i}) - B(y^{*i})^T \beta^{*i}]\}$$
$$+ \sum_{j \in J_0} v_j^* [\nabla G_j(x^*) + C_j^T \gamma^{*j}] + \sum_{k \in K_0} w_k^* \nabla H_k(x^*)\Big\}\Big) < - \sum_{i \in I_+} u_i^* \bar{\rho}_i(\bar{x}, x^*) \|\theta(\bar{x}, x^*)\|^2. \quad (5.5)$$

As seen in the proof of Theorem 4.1, our assumptions in (ii) lead to

$$\mathcal{F}\Big(\bar{x}, x^*; b(\bar{x}, x^*) \sum_{t=1}^{m} \Big\{ \sum_{j \in J_t} v_j^* [\nabla G_j(x^*) + C_j^T \gamma^{*j}] + \sum_{k \in K_t} w_k^* \nabla H_k(x^*)\Big\}\Big)$$
$$\leq - \sum_{t=1}^{m} \tilde{\rho}_t(\bar{x}, x^*) \|\theta(\bar{x}, x^*)\|^2,$$

which when combined with (5.1), results into

$$\mathcal{F}\Big(\bar{x}, x^*; b(\bar{x}, x^*)\Big\{ \sum_{i=1}^{p} u_i^* \{\nabla f(x^*, y^{*i}) + A(y^{*i})^T \alpha^{*i} - \lambda^* [\nabla g(x^*, y^{*i}) - B(y^{*i})^T \beta^{*i}]\}$$
$$+ \sum_{j \in J_0} v_j^* [\nabla G_j(x^*) + C_j^T \gamma^{*j}] + \sum_{k \in K_0} w_k^* \nabla H_k(x^*)\Big\}\Big) \geq \sum_{t=1}^{m} \tilde{\rho}_t(\bar{x}, x^*) \|\theta(\bar{x}, x^*)\|^2.$$

In view of (iii), this inequality contradicts (5.5). Hence, x^* is an optimal solution of (P).

(b) - (g): The proofs are similar to that of part (a). □

Each one of the twenty-two sets of conditions given in Theorems 5.1 and 5.2, and in their modified versions obtained by replacing (4.1) with (4.10), can be viewed as a family of sufficient optimality conditions whose members can easily be identified by appropriate choices of the partitioning sets J_μ and $K_\mu,\ \mu \in \underline{m} \cup \{0\}$.

In the remainder of this section, we present another collection of sufficiency results which are somewhat different from those stated in Theorems 5.1 and 5.2. These results are formulated by utilizing a partition of $\underline{p}$ in addition to those of $\underline{q}$ and $\underline{r}$, and by placing appropriate generalized $(\mathcal{F}, b, \phi, \rho, \theta)$-univexity requirements on certain combinations of the problem functions.

Let $\{I_0, I_1, \ldots, I_\ell\}$ be a partition of $\underline{p}$ such that $L = \{0, 1, 2, \ldots, \ell\} \subseteq M = \{0, 1, \ldots, m\}$, and let the function $\Pi_t(\cdot, \bar{y}, \lambda, u, v, w, \alpha, \beta, \gamma)\colon X \to \mathbb{R}$ be defined, for fixed $\bar{y}, \lambda, u, v, w, \alpha, \beta$, and γ, by

$$\Pi_t(x, \bar{y}, \lambda, u, v, w, \alpha, \beta, \gamma) = \sum_{i \in I_t} u_i \{f(x, y^i) + \langle \alpha^i, A(y^i)x\rangle - \lambda[g(x, y^i)$$
$$- \langle \beta^i, B(y^i)x\rangle]\} + \sum_{j \in J_t} v_j [G_j(x) + \langle \gamma^j, C_j x\rangle] + \sum_{k \in K_t} w_k H_k(x), \quad t \in L.$$

Theorem 5.3. *Let $x^* \in \mathbb{F}$, let $\lambda^* = \varphi(x^*) \geq 0$, and assume that the functions $f(\cdot, y)$, $g(\cdot, y)$, $y \in Y$, G_j, $j \in \underline{q}$, and H_k, $k \in \underline{r}$, are differentiable at x^*, and that there exist $(p, \bar{y}^*, u^*,$*

$\alpha^*, \beta^*) \in \mathbb{K}$, $v^* \in \mathbb{R}^q_+$, $w^* \in \mathbb{R}^r$, *and* $\gamma^{*j} \in \mathbb{R}^{n_j}$, $j \in \underline{q}$, *such that* (4.1) - (4.5) *hold. Assume, furthermore, that any one of the following seven sets of hypotheses is satisfied:*

(a) (i) *for each* $t \in L$, $\Pi_t(\cdot, \bar{y}^*, \lambda^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ *is strictly* $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$ *-pseudounivex at* x^*, ϕ_t *is increasing, and* $\phi_t(0) = 0$*;*

(ii) *for each* $t \in M \setminus L, \Lambda_t(\cdot, v^*, w^*, \gamma^*)$ *is* $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$*-quasiunivex at* x^*, ϕ_t *is increasing, and* $\phi_t(0) = 0$*;*

(iii) $\sum\limits_{t \in M} \rho_t(x, x^*) \geq 0$ *for all* $x \in \mathbb{F}$*;*

(b) (i) *for each* $t \in L$, $\Pi_t(\cdot, \bar{y}^*, \lambda^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ *is* $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$*-quasiunivex at* x^*, ϕ_t *is increasing, and* $\phi_t(0) = 0$*;*

(ii) *for each* $t \in M \setminus L$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ *is strictly* $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$*-pseudounivex at* x^*, ϕ_t *is increasing, and* $\phi_t(0) = 0$*;*

(iii) $\sum\limits_{t \in M} \rho_t(x, x^*) \geq 0$ *for all* $x \in \mathbb{F}$*;*

(c) (i) *for each* $t \in L$, $\Pi_t(\cdot, \bar{y}^*, \lambda^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ *is* $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$*-quasiunivex at* x^*, ϕ_t *is increasing, and* $\phi_t(0) = 0$*;*

(ii) *for each* $t \in M \setminus L$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ *is* $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$*-quasiunivex at* x^*, ϕ_t *is increasing, and* $\phi_t(0) = 0$*;*

(iii) $\sum\limits_{t \in M} \rho_t(x, x^*) > 0$ *for all* $x \in \mathbb{F}$*;*

(d) (i) *for each* $t \in L_1$, $\Pi_t(\cdot, \bar{y}^*, \lambda^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ *is strictly* $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$ *-pseudounivex at* x^*, *for each* $t \in L_2$, $\Pi_t(\cdot, \bar{y}^*, \lambda^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ *is* $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$*-quasiunivex at* x^*, *and for each* $t \in L$, ϕ_t *is increasing and* $\phi_t(0) = 0$, *where* $\{L_1, L_2\}$ *is a partition of* L*;*

(ii) *for each* $t \in M \setminus L$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ *is strictly* $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$*-pseudounivex at* x^*, ϕ_t *is increasing, and* $\phi_t(0) = 0$*;*

(iii) $\sum\limits_{t \in M} \rho_t(x, x^*) \geq 0$ *for all* $x \in \mathbb{F}$*;*

(e) (i) *for each* $t \in L_1 \neq \emptyset$, $\Pi_t(\cdot, \bar{y}^*, \lambda^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ *is strictly* $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$*-pseudounivex at* x^*, *for each* $t \in L_2$, $\Pi_t(\cdot, \bar{y}^*, \lambda^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ *is* $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$*-quasiunivex at* x^*, *and for each* $t \in L$, ϕ_t *is increasing and* $\phi_t(0) = 0$, *where* $\{L_1, L_2\}$ *is a partition of* L*;*

(ii) *for each* $t \in M \setminus L, \Lambda_t(\cdot, v^*, w^*, \gamma^*)$ *is* $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$*-quasiunivex at* x^*, ϕ_t *is increasing, and* $\phi_t(0) = 0$*;*

(iii) $\sum\limits_{t \in M} \rho_t(x, x^*) \geq 0$ *for all* $x \in \mathbb{F}$*;*

(f) (i) *for each* $t \in L$, $\Pi_t(\cdot, \bar{y}^*, \lambda^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ *is* $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$*-quasiunivex at* x^*, ϕ_t *is increasing, and* $\phi_t(0) = 0$*;*

(ii) *for each* $t \in (M \setminus L)_1 \neq \emptyset$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ *is strictly* $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$*-pseudounivex at* x^*, *for each* $t \in (M \setminus L)_2$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ *is* $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$*-quasiunivex at* x^*, *and for each* $t \in L$, ϕ_t *is increasing and* $\phi_t(0) = 0$, *where* $\{(M \setminus L)_1, (M \setminus L)_2\}$ *is a partition of* $M \setminus L$*;*

(iii) $\sum\limits_{t \in M} \rho_t(x, x^*) \geq 0$ *for all* $x \in \mathbb{F}$*;*

(g) (i) *for each* $t \in L_1$, $\Pi_t(\cdot, \bar{y}^*, \lambda^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ *is strictly* $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$ *-pseudounivex at* x^*, *for each* $t \in L_2$, $\Pi_t(\cdot, \bar{y}^*, \lambda^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)$ *is* $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$*-quasiunivex at* x^*, *and for each* $t \in L$, ϕ_t *is increasing and* $\phi_t(0) = 0$, *where* $\{L_1, L_2\}$ *is a partition of* L;

(ii) *for each* $t \in (M \setminus L)_1$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ *is strictly* $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$*-pseudounivex at* x^*, *for each* $t \in (M \setminus L)_2$, $\Lambda_t(\cdot, v^*, w^*, \gamma^*)$ *is* $(\mathcal{F}, b, \phi_t, \rho_t, \theta)$*-quasiunivex at* x^*, *and for each* $t \in M \setminus L$, ϕ_t *is increasing and* $\phi_t(0) = 0$, *where* $\{(M \setminus L)_1, (M \setminus L)_2\}$ *is a partition of* $M \setminus L$;

(iii) $\sum\limits_{t \in M} \rho_t(x, x^*) \geq 0$ *for all* $x \in \mathbb{F}$;

(iv) $L_1 \neq \emptyset$, $(M \setminus L)_1 \neq \emptyset$, *or* $\sum\limits_{t \in M} \rho_t(x, x^*) > 0$ *for all* $x \in \mathbb{F}$.

Then x^* *is an optimal solution of* (P).

Proof. (a): Suppose to the contrary that x^* is not an optimal solution of (P). As seen in the proof of Theorem 5.2, this supposition leads to the inequalities

$$f(\bar{x}, y^{*i}) + \|A(y^{*i})\bar{x}\|_a - \lambda^*[g(\bar{x}, y^{*i}) - \|B(y^{*i})\bar{x}\|_b] < 0, \quad i \in \underline{p},$$

for some $\bar{x} \in \mathbb{F}$. Since $u^* \geq 0$, we see that for each $t \in L$,

$$\sum_{i \in I_t} u_i^* \{f(\bar{x}, y^{*i}) + \|A(y^{*i})\bar{x}\|_a - \lambda^*[g(\bar{x}, y^{*i}) - \|B(y^{*i})\bar{x}\|_b]\} \leq 0. \tag{5.6}$$

Now using this inequality, we see that

$$\begin{aligned}
&\Pi_t(\bar{x}, \bar{y}^*, \lambda^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*) \\
&= \sum_{i \in I_t} u_i^* \{f(\bar{x}, y^{*i}) + \langle \alpha^{*i}, A(y^{*i})\bar{x}\rangle - \lambda^*[g(\bar{x}, y^{*i}) - \langle \beta^{*i}, B(y^{*i})\bar{x}\rangle]\} + \\
&\quad \sum_{j \in J_t} v_j^*[G_j(\bar{x}) + \langle \gamma^{*j}, C_j\bar{x}\rangle] + \sum_{k \in K_t} w_k^* H_k(\bar{x}) \\
&\leq \sum_{i \in I_t} u_i^* \{f(\bar{x}, y^{*i}) + \|\alpha^{*i}\|_a^* \|A(y^{*i})\bar{x}\|_a - \lambda^*[g(\bar{x}, y^{*i}) \\
&\quad - \|\beta^{*i}\|_b^* \|B(y^{*i})\bar{x}\|_b]\} + \sum_{j \in J_t} v_j^*[G_j(\bar{x}) + \|\gamma^{*j}\|_{c(j)}^* \|C_j\bar{x}\|_{c(j)}] \\
&\quad \text{(by Lemma 4.1 and the feasibility of } \bar{x}) \\
&\leq \sum_{i \in I_t} u_i^* \{f(\bar{x}, y^{*i}) + \|A(y^{*i})\bar{x}\|_a - \lambda^*[g(\bar{x}, y^{*i}) - \|B(y^{*i})\bar{x}\|_b]\} + \\
&\quad \sum_{j \in J_t} v_j^*[G_j(\bar{x}) + \|C_j\bar{x}\|_{c(j)}] \ \text{ (by (4.4) and (4.5))} \\
&\leq \sum_{i \in I_t} u_i^* \{f(\bar{x}, y^{*i}) + \|A(y^{*i})\bar{x}\|_a - \lambda^*[g(\bar{x}, y^{*i}) - \|B(y^{*i})\bar{x}\|_b]\} \\
&\quad \text{(by the feasibility of } \bar{x}) \\
&\leq 0 \quad \text{(by (5.6))}
\end{aligned}$$

$$= \sum_{i\in I_t} u_i^*\{f(x^*, y^{*i}) + \langle \alpha^{*i}, A(y^{*i})x^*\rangle - \lambda^*[g(x^*, y^{*i}) - \langle \beta^{*i} B(y^{*i})x^*\rangle]\}+$$
$$\sum_{j\in J_t} v_j^*[G_j(x^*) + \langle \gamma^{*j}, C_j x^*\rangle] + \sum_{k\in K_t} w_k^* H_k(x^*)$$
(by (4.2), (4.3), and the feasibility of x^*)
$$= \Pi_t(x^*, \bar{y}^*, \lambda^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*),$$

and hence

$$\phi_t\big(\Pi_t(\bar{x}, \bar{y}^*, \lambda^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*) - \Pi_t(x^*, \bar{y}^*, \lambda^*, u^*, v^*, w^*, \alpha^*, \beta^*, \gamma^*)\big) \le 0,$$

which in view of (i) implies that

$$\mathcal{F}\Big(\bar{x}, x^*; b(\bar{x}, x^*)\Big\{ \sum_{i\in I_t} u_i^*\{\nabla f(x^*, y^{*i}) + A(y^{*i})^T \alpha^{*i} - \lambda^*[\nabla g(x^*, y^{*i})$$
$$-B(y^{*i})^T \beta^{*i}]\} + \sum_{j\in J_t} v_j^*[\nabla G_j(x^*) + C_j^T \gamma^{*j}] + \sum_{k\in K_t} w_k^* \nabla H_k(x^*)\Big\}\Big) < -\rho_t(\bar{x}, x^*)\|\theta(\bar{x}, x^*)\|^2.$$

Summing over $t \in L$, and using the sublineariy of $\mathcal{F}(\bar{x}, x^*; \cdot)$, we obtain

$$\mathcal{F}\Big(\bar{x}, x^*; b(\bar{x}, x^*)\Big\{ \sum_{i=1}^{p} u_i^*\{\nabla f(x^*, y^{*i}) + A(y^{*i})^T \alpha^{*i} - \lambda[\nabla g(x^*, y^{*i})$$
$$- B(y^{*i})^T \beta^{*i}]\} + \sum_{t\in L}\Big\{ \sum_{j\in J_t} v_j^*[\nabla G_j(x^*) + C_j^T \gamma^{*j}] + \sum_{k\in K_t} w_k^* \nabla H_k(x^*)\Big\}\Big\}\Big)$$
$$< -\sum_{t\in L} \rho_t(\bar{x}, x^*)\|\theta(\bar{x}, x^*)\|^2. \quad (5.7)$$

As shown in the proof of Theorem 4.1, for each $t \in M\backslash L$, $\Lambda_t(\bar{x}, v^*, w^*, \gamma^*) \le \Lambda_t(x^*, v^*, w^*, \gamma^*)$, and so $\phi_t\big(\Lambda_t(\bar{x}, v^*, w^*, \gamma^*) - \Lambda_t(x^*, v^*, w^*, \gamma^*)\big) \le 0$, which in view of (ii) implies that

$$\mathcal{F}\Big(\bar{x}, x^*; b(\bar{x}, x^*)\Big\{ \sum_{j\in J_t} v_j^*[\nabla G_j(x^*) + C_j^T \gamma^{*j}] + \sum_{k\in K_t} w_k^* \nabla H_k(x^*)\Big\}\Big) \le -\rho_t(\bar{x}, x^*)\|\theta(\bar{x}, x^*)\|^2.$$

Summing over $t \in M\backslash L$ and using the sublinearity of $\mathcal{F}(\bar{x}, x^*; \cdot)$, we get

$$\mathcal{F}\Big(\bar{x}, x^*; b(\bar{x}, x^*) \sum_{t\in M\backslash L}\Big\{ \sum_{j\in J_t} v_j^*[\nabla G_j(x^*) + C_j^T \gamma^{*j}] + \sum_{k\in K_t} w_k^* \nabla H_k(x^*)\Big\}\Big)$$
$$\le -\sum_{t\in M\backslash L} \rho_t(\bar{x}, x^*)\|\theta(\bar{x}, x^*)\|^2. \quad (5.8)$$

Now combining (5.7) and (5.8) and using the sublinearity of $\mathcal{F}(\bar{x}, x^*; \cdot)$ and (iii), we obtain

$$\mathcal{F}\Big(\bar{x}, x^*; b(\bar{x}, x^*)\Big\{ \sum_{i=1}^{p} u_i^*\{\nabla f(x^*, y^{*i}) + A(y^{*i})^T \alpha^{*i} - \lambda^*[\nabla g(x^*, y^{*i}) - B(y^{*i})^T \beta^{*i}]\} +$$
$$\sum_{j=1}^{q} v_j^*[\nabla G_j(x^*) + C_j^T \gamma^{*j}] + \sum_{k=1}^{r} w_k^* \nabla H_k(x^*), \eta(\bar{x}, x^*)\Big\}\Big) < -\sum_{t\in M} \rho_t(\bar{x}, x^*)\|\theta(\bar{x}, x^*)\|^2 \le 0,$$

which contradicts (4.1). Therefore, x^* is an optimal solution of (P).

(b) - (g): The proofs are similar to that of part (a). □

The fourteen families of sufficiency results stated in Theorem 5.3, and in its modified version obtained by replacing (4.1) with (4.10), contain a very large number of special cases and variants which can readily be identified by appropriate choices of the partitioning sets $I_0, I_1, \ldots, I_\ell$, and the sets L and M.

6. CONCLUDING REMARKS

Using a Dinkelbach-type [8] parametric approach, in this paper we have established a set of necessary optimality conditions and numerous sets of sufficient optimality criteria for a continuous minmax fractional programming problem containing arbitrary norms (and square roots of positive semidefinite quadratic forms). In formulating these criteria, we have exploited the structure of the subdifferentials of the arbitrary norms in conjunction with the basic properties of various types of generalized $(\mathcal{F}, b, \phi, \rho, \theta)$-univex functions. Since all the optimality conditions and especially the sufficiency results proved for the prototype problem (P) can easily be modified and restated for each one of the seventeen special cases of (P) designated as $(P1)$ - $(P17)$ in Section 1, they collectively subsume a truly vast number of sufficiency results previously obtained for several classes of nonlinear programming problems by an assortment of *ad hoc* methods. Furthermore, the style of presentation adopted in this paper as well as the main results derived here will prove useful in studying other related categories of nonlinear programming problems and utilizing other types of generalized convexity concepts. In particular, the results of the present study can be extended to the following semiinfinite version of (P):

$$\text{Minimize} \quad \max_{y \in Y} \frac{f(x,y) + \|A(y)x\|_a}{g(x,y) - \|B(y)x\|_b}$$

subject to

$$\tilde{G}_j(x,t) + \|C_j(t)x\|_{c(j)} \leq 0 \ \text{ for all } t \in T_j, \ \ j \in \underline{q},$$

$$\tilde{H}_k(x,s) = 0 \ \text{ for all } s \in S_k, \ \ k \in \underline{r},$$

$$x \in X,$$

where X, Y, f, g, $A(y)$, $B(y)$, $\|\cdot\|_a$, and $\|\cdot\|_b$ are as defined in the description of (P), T_j and S_k are compact subsets of complete metric spaces, for each $j \in \underline{q}$, $\tilde{G}_j(\cdot, t)$ is a real-valued function defined on X for all $t \in T_j$, for each $k \in \underline{r}$, $\tilde{H}_k(\cdot, s)$ is a real-valued function defined on X for all $s \in S_k$, for each $j \in \underline{q}$ and $k \in \underline{r}$, $\tilde{G}_j(x, \cdot)$ and $\tilde{H}_k(x, \cdot)$ are continuous real-valued functions defined, respectively, on T_j and S_k for all $x \in X$, and for each $j \in \underline{q}$ and $t \in T_j$, $C_j(t)$ is an $n_j \times n$ matrix and $\|\cdot\|_{c(j)}$ is an arbitrary norm on $\mathbb{R}^{n_j}$.

We shall investigate this semiinfinite programming problem in subsequent papers.

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