J. Adv. Math. Stud.
Vol. **5**(2012), No. 2, 13-19
http://journal.fairpartners.ro

SEMILLATICE-ORDERED SEMIGROUPS WITH APARTNESS REPRESENTATION PROBLEM

DANIEL A. ROMANO

Abstract. Semilattice-ordered semigroups are important algebraic structures. They are ordered semigroups under partial order and ordered semigroups under anti-order. In this article we analyze constructive version of representation problem of semilattice-ordered semigroups with apartness in two theorems. Difference between the classical case and the constructive case is in using the Constructive logic instead the Classical logic. Besides, since algebraic structures here are supplied with diverse relations, we especially analyze how it affects the representation problem.

1. INTRODUCTION, PRELIMINARIES AND EXAMPLES

1.1. **The setting.** Our setting is Bishop's constructive mathematics ([3], [4], [5], [6], [8] and [15]), mathematics developed with Constructive logic (or Intuitionistic logic) - logic without the Law of Excluded Middle $P \vee \neg P$. We have to note that 'the crazy axiom' $\neg P \Longrightarrow (P \Longrightarrow Q)$ is included in the Constructive logic. Precisely, in Constructive logic the 'Double Negation Law' $P \Longleftrightarrow \neg\neg P$ does not hold, but the following implication $P \Longrightarrow \neg\neg P$ holds even in Minimal logic. In Constructive logic, 'Weak Law of Excluded Middle' $\neg P \vee \neg\neg P$ does not hold as well. It is interesting, in Constructive logic the following deduction principle $A \vee B, \neg A \vdash B$ holds, but this is impossible to prove without 'the crazy axiom'.

1.2. **Sets with diversity.** This investigation is in Bishop's constructive algebra in sense of papers [9], [11], [12], [13] and [14], and books [8] and [15] (Chapter 8: Algebra). Let $(A, =, \neq)$ be a constructive set. The diversity relation "$\neq$" is a binary relation on A, which satisfies the following properties:

$$\neg(x \neq x),\ x \neq y \Longrightarrow y \neq x,\ (x \neq y \wedge y = z) \Longrightarrow x \neq z.$$

If it satisfies the following condition

$$x \neq z \Longrightarrow (\forall y \in A)(x \neq y \vee y \neq z),$$

it is called *apartness* (A. Heyting).

For subsets X and Y of A we say that set X is set-set apartness from Y, and it is denoted by $X \bowtie Y$, if and only if $(\forall x \in X)(\forall y \in Y)(x \neq y)$. We set $x \bowtie Y$ instead of $\{x\} \bowtie Y$, and, of course, $x \neq y$ instead of $\{x\} \bowtie \{y\}$. With $X^C = \{x \in A : x \bowtie X\}$ we denote apartness complement of X in A. So, "$\bowtie$" is a relation between pairs of subsets of A (for more details, see [6]).

Received: June 05, 2011 *Revised:* February 27, 2012.
2010 *Mathematics Subject Classification*: 03F65, 06F05
Key words and phrases: Constructive mathematics, semigroup with apartness, semilattice-ordered semigroup.

©2012 Fair Partners Team for the Promotion of Science & Fair Partners Publishers

Let Y be a subset of $(X, =, \neq)$. We say that it is detachable if and only if $(\forall x)$ $(x \in Y \vee x \bowtie Y)$.

For a function $f\colon (X, =, \neq) \longrightarrow (Y, =, \neq)$ we say that it is a strongly extensional if and only if $(\forall a, b \in X)(f(a) \neq f(b) \Longrightarrow a \neq b)$.

For undefined notions and notations we referred to books [2], [3], [4], [5], [6] and for items of Constructive Algebra we referred to books [8] and [15] and to the author's papers [9]-[14].

1.3. **Semilattice-ordered semigroups.** Following articles [1], [9] and [13], algebraic structure $(A, =, \neq, \cdot, \otimes)$ is called a (strong) semilattice-ordered semigroup if:

(i) $(A, =, \neq, \cdot)$ is a semigroup, where the semigroup operation "$\cdot$" is strongly extensional in the following way

$$(\forall a, b, c \in A)((ac \neq bc \vee ca \neq cb) \Longrightarrow a \neq b);$$

(ii) $(A, =, \neq, \otimes)$ is a semilattice, i.e. $(A, \otimes)$ is a commutative semigroup with $(\forall x \in A)$ $(x \otimes x = x)$, where the semigroup operation is strongly extensional:

$$(\forall a, b, c \in A)((a \otimes c \neq b \otimes c \vee c \otimes a \neq c \otimes b) \Longrightarrow a \neq b);$$

(iii) $(\forall a, b, c \in A)((a(b \otimes c) = ab \otimes ac) \wedge ((a \otimes b)c = ac \otimes bc))$.

The following proposition shows significance of semilattice-ordered semigroup:

Proposition 1.1 ([1], [2], [9] or [13])**.** *If $(A, =, \neq, \cdot, \otimes)$ is a semilattice-ordered semigroup and we define, for any a, b of A, $a \leq b$ if and only if $a \otimes b = b$, it is known that the structure $(A, =, \neq, \cdot, \leq)$ is an ordered semigroup.*

Since in Constructive logic the 'Law of Excluded Middle' is not valid, in Bishop's constructive algebra the following relation is also interesting: a relation symmetric to ordered relation $\leq$. A relation θ on S is anti-order ([11], [12], [14]) on set S if and only if

$$\theta \subseteq \neq ,$$
$$(\forall x, y, z \in S)((x, z) \in \theta \Longrightarrow ((x, y) \in \theta \vee (y, z) \in \theta)),$$
$$(\forall x, y \in S)(x \neq y \Longrightarrow ((x, y) \in \theta \vee (y, x) \in \theta)), \text{ (linearity) and}$$
$$(\forall x, y, z \in S)(((xz, yz) \in \theta \Longrightarrow (x, y) \in \theta) \wedge ((zx, zy) \in \theta \Longrightarrow (x, y) \in \theta)).$$

System $(A, =, \neq, \cdot, \theta)$ is ordered semigroup under anti-order if $(A, =, \neq, \cdot)$ is a semigroup where the semigroup operation is strongly extensional and relation θ is an anti-order relation on $(A, =, \neq, \cdot)$.

In the following assertion we show that semilattice-ordered semigroup is relevant once more. This suggests that there exists interest in investigation of ordered semigroup under anti-order.

Proposition 1.2 ([9], Lemma or [13], Lemma 2.2)**.** *If $(A, =, \neq, \cdot, \otimes)$ is a semilattice-ordered semigroup and we define, for any a, b of A, $a\theta b \Longleftrightarrow a \otimes b \neq b$, then the structure $(A, =, \neq, \cdot, \theta)$ is an ordered semigroup under anti-order θ.*

1.4. **The results.** The main goal of this paper is to reopen semigroup representation problem. This time, for semilattice-ordered semigroup with diversity or apartness we give two results: First result is concerned for any semilattice-ordered semigroup with apartness. The second one is concerned on semigroup $(\wp(S), =, \neq, \cdot, \cup)$ because the diversity relation $\neq$ is not an apartness in this case. Here $\wp(A)$ is the set of all subsets of a set A. In the subsection 1.5 we give several examples. In Example 1 we construct free semigroup with apartness generated by given set with apartness. In Example 2 we show

that structure $(End_{\neq}(A,=_A,\neq_A,\otimes),=,\neq,\circ,\diamond)$ of all strongly extensional endomorphisms of semigroup $(A,=_A,\neq_A,\cdot)$ with apartness is a semilattice-ordered semigroup with apartness. Structure $(Rel(S^1),=_2,\neq_2,\circ,\cup)$ is a semilattice-ordered semigroup where the relation $\neq_2$ is diversity relation shown in Example 4. In section 2, it is shown (Theorem 2.1) that any (strong) semilattice-ordered semigroup with apartness and with strongly extensional operations is isomorphic to a substructure of a semilattice-ordered semigroup $(End_{\neq}(A,),=,\neq,\circ,\diamond)$ generated in Example 3. The connection between the semigroups $(\wp(S),=,\neq,\cdot,\otimes)$ and $(Rel(S^1),=,\neq,\circ,\diamond)$ is given in this section (Theorem 2.2).

1.5. **Examples.** As first, we give a construction of free semigroup with apartness generated by set $(X,=,\neq)$ following the concept expressed in articles [7] and [10].

Example 1.1. ([13], Example 3.2) Let $(X,=,\neq)$ be a set with apartness. We form the following class X^+ of all strictly finite sequences of elements of X

$$x^+ \in X^+ \Longleftrightarrow (n_x \in \mathbb{N})(\exists f_x)(f_x\colon \{1,2,\ldots,n_x\} \longrightarrow x^+)$$

with

$$(\forall i \in \{1,2,\ldots,n_x\})(f_x(i) \in X).$$

As usual the concatenation of x^+ and y^+ is denoted by $x^+ \circ y^+$. If $x^+ = (f_x(1),\ldots,f_x(n_x))$ and $y^+ = (f_y(1),\ldots,f_y(n_y))$, then

$$n_{xy} = n_x + n_y \text{ and}$$
$$i \in \{1,2,\ldots,n_x\} \Longrightarrow f_{xy}(i) = f_x(i),$$
$$i = n_x + j\,(j \in \{1,2,\ldots,n_y\}) \Longrightarrow f_{xy}(i) = f_y(j),$$

i.e.

$$x^+ \circ y^+ = (f_x(1),\ldots,f_x(n_x),f_y(1),\ldots,f_y(n_y)).$$

On the class X^+, we define

$$x^+ =_1 y^+ \Longleftrightarrow (n_x = n_y \wedge f_x = f_y) \text{ and}$$
$$x^+ \neq_1 y^+ \Longleftrightarrow (\neg(n_x = n_y) \vee f_x \neq f_y).$$

Finally, as shown in [13], the structure $(X+,=_1,\neq_1,\circ)$ is a semigroup with apartness.

As second, we present certain basic and natural examples of semilattice-ordered semigroups.

Example 1.2. ([13], Example 3.3) Let $(A,=_A,\neq_A,\otimes)$ be a semilattice with strongly extensional operation $\otimes$. With $End_{\neq}(A,=_A,\neq_A,\otimes)$ we mark the set of all strongly extensional endomorphisms of semilattice $(A,=_A,\neq_A,\otimes)$ with joint and operation of composition as in Example 2. The inclusion $End_{\neq}(A,=_A,\neq_A,\otimes) \subseteq End(A,=_A,\neq_A,\otimes)$ holds. Equality $=$ and apartness $\neq$ on $End_{\neq}(A,=_A,\neq_A,\otimes)$ we defined by the usual way:

$$\alpha = \beta \Longleftrightarrow (\forall x \in A)(\alpha(x) =_A \beta(x)),\ \alpha \neq \beta \Longleftrightarrow (\exists\, x \in A)(\alpha(x) \neq_A \beta(x)).$$

Example 1.3. ([2], XIV.1, Example 2) Let S be a semigroup. We put, for any $X \in \wp(S)$ and $Y \in \wp(S)$ $XY = \{xy | x \in X \wedge y \in Y\}$. Then $(\wp(S),\cdot,\cup)$ is a semilattice-ordered semigroups. Here, as usual, $\cup$ denotes the set-theoretical union.

Let us note that if the set S is supplied by apartness, relation $\neq_1$ on $\wp(S)$, defined by the natural way: $X \neq_1 Y \Longleftrightarrow (\exists\, x \in X)(x \bowtie Y) \vee (\exists\, y \in Y)(y \bowtie X)$ is a diversity relation but it is not apartness, in general.

Example 1.4. ([13], Example 3.4) Let $(A, =, \neq)$ be a set with apartness. Then the set $Rel(A)$ of all binary relations on A with the operations of composition "$\circ$" and union $\cup$, with equality $=_2$ and diversity $\neq_2$ defined by the usual way:

$$\alpha =_2 \beta \Longleftrightarrow (\forall x, y \in A)((x, y) \in \alpha \Longleftrightarrow (x, y) \in \beta),$$

$$\alpha \neq_2 \beta \Longleftrightarrow$$

$$(\exists\, x, y \in A)((x, y) \in \alpha \wedge (x, y) \bowtie \beta) \vee (\exists\, x, y \in A)((x, y) \bowtie \alpha \wedge (x, y) \in \beta),$$

is a semilattice ordered semigroup. Let us note again that the relation $\neq_2$ is a diversity but it is not apartness, in general case. The composition "$\circ$" of relations is a strongly extensional operation.

2. REPRESENTATIONS

The importance of the given examples follows from the next theorems.

Theorem 2.1. *Every (strong) semilattice-ordered semigroup with apartness and with strongly extensional operations is isomorphic to a substructure of a semilattice-ordered semigroup* $(End_{\neq}(A,), =, \neq, \circ, \diamond)$.

Proof. Let $(A, =, \neq, \cdot, \otimes)$ be a semilattice-ordered semigroup. We add to A a new element $e\,(\bowtie A)$, denote $A^e = A \cup \{e\}$, we also extend the operations "$\cdot$" and "$\otimes$" to A^e as follows:

$$(\forall a \in A)(e \cdot e = e \cdot a = a \cdot e = e,\ e \otimes e = e,\ e \otimes a = a \otimes e = a).$$

As it is well-known, $(A^e, =, \neq, \cdot, \otimes)$ is a semilattice-ordered semigroup.

Define a binary operation $\otimes$ on the set $\{0, 1\}\,(0 \neq 1)$ in the following way:

$$0 \otimes 0 = 0,\ 1 \otimes 1 = 1,\ 0 \otimes 1 = 1 \otimes 0 = 1.$$

The operation $\otimes$ is strongly extensional and, clearly, $(\{0, 1\}, \otimes)$ is a semilattice. Put

$$(B, =, \neq, \odot) = (A^e, \otimes) \times (\{0, 1\}, \otimes),$$

where

$$(x, i) = (y, j) \Longleftrightarrow (x = y \wedge i = j) \text{ and}$$
$$(x, i) \neq (y, j) \Longleftrightarrow (x \neq y \vee i \neq j).$$

It is easy to see that the operation $\odot$, defined by

$$(x, i) \odot (y, j) = (x \otimes y, i \otimes j),$$

is strongly extensional. Indeed: Let (x, i), (y, j), (a, k) and (b, h) be arbitrary elements. Suppose that $(x, i) \odot (y, j) \neq (a, k) \odot (b, h)$. Then $(x \otimes y, i \otimes j) \neq (a \otimes b, k \otimes h)$, i.e. $x \otimes y \neq a \otimes b \wedge i \otimes j \neq k \otimes h$. Thus, $(x \neq a \vee y \neq b)$ and $(i \neq k \vee j \neq h)$. Therefore, $(x, i) \neq (a, k)$ or $(y, j) \neq (b, h)$.

For any $a \in A$, define a mapping $F(a)\colon B \longrightarrow B$ as follows: Let $x \in A^e$, $i \in \{0, 1\}$. We put

$$F(a)((x, 0)) = (ax, 0),\ F(a)((x, 1)) = (ax \otimes a, 0).$$

We shall show: $F(a) \in End(B, \odot)$ for any $a \in A$.

Let $x, y \in A^e$, and $i, j \in \{0, 1\}$. We distinguish four cases:

(a) If $i = j = 0$ then

$$F(a)((x,0)\odot(y,0)) = F(a)((x\otimes y,0)) = (ax\otimes ay,0) \text{ and}$$
$$F(a)((x,0))\odot F(a)((y,0)) = (ax,0)\odot(ay,0) = (ax\otimes ay,0).$$

(b) If $i=0$, $j=1$ then

$$F(a)((x,0)\odot(y,1)) = F(a)((x\otimes y,1)) = (ax\otimes ay\otimes a,0) \text{ and}$$
$$F(a)((x,0))\odot F(a)((y,1)) = (ax,0)\odot(ay\otimes a,0) = (ax\otimes ay\otimes a,0).$$

(c) For $i=1, j=0$ proof is similar to (b).
(d) If $i=j=1$, then

$$F(a)((x,1)\odot(y,1)) = F(a)((x\otimes y,1)) = (ax\otimes ay\otimes a,0) \text{ and}$$
$$F(a)((x,1))\odot F(a)((y,1)) = (ax\otimes a,0)\odot(ay\otimes a,0) = (ax\otimes ay\otimes a,0).$$

Now, we know that F maps A to $End(B,\odot)$. To finish the proof of this theorem it remains to show that F is an injective and embedding strongly extensional homomorphism from $(A,=,\neq,\cdot,\otimes)$ to $(End(B,\odot),\circ,\diamond)$.

(i) F is a strongly extensional. Let $a,b\in A$. We want to show that $F(a)\neq F(b) \Longrightarrow a\neq b$. Let $x\in A^e$, $i\in\{0,1\}$ such that $F(a)((x,i))\neq F(b)((x,i))$.

For $i=0$, we have $F(a)((x,i) = (ax,0)$, $F(b)((x,i)) = (bx,0)$ and from $(ax,0)\neq(bx,0)$ follows $a\neq b$.

For $i=1$, we have $F(a)((x,1)) = (ax\otimes a,0)$, $F(b)((x,i)) = (bx\otimes b,0)$ and from $(ax\otimes a,0)\neq(bx\otimes b,0)$ follows $ax\otimes a\neq bx\otimes b$. Thus $ax\neq bx\vee a\neq b$. So, $a\neq b$, in both sides. Therefore, mapping F is strongly extensional.

(ii) F is a homomorphism: Let $a,b\in A$. We want to show that $F(ab)=F(a)\circ F(b)$. Let $x\in A^e$, $i\in\{0,1\}$.
For $i=0$ we have

$$F(ab)((x,0)) = (abx,0),$$
$$(F(a)\circ F(b))((x,0)) = F(a)(F(b)((x,0))) = F(a)((bx,0)) = (abx,0),$$

and for $i=1$ we have

$$F(ab)((x,1)) = (abx\otimes ab,0),$$
$$(F(a)\circ F(b))((x,1)) = F(a)(F(b)((x,1))) = F(a)((bx\otimes b,0)) = (abx\otimes ab,0).$$

Further we want to show that $F(a\otimes b) = F(a)\diamond F(b)$. Let $x\in A^e$, $i\in\{0,1\}$. For $i=0$ we have

$$F(a\otimes b)((x,0)) = ((a\otimes b)x,0),$$
$$(F(a)\diamond F(b))((x,0)) = F(a)((x,0))\odot F(b)((x,0)) = (ax,0)\odot(bx,0) = ((a\otimes b)x,0),$$

and for $i=1$ we have

$$F(a\otimes b)((x,1)) = ((a\otimes b)x\otimes a\otimes b,0),$$
$$(F(a)\diamond F(b))((x,1)) = F(a)((x,1))\odot F(b)((x,1)) = (ax\otimes a,0)\odot(bx\otimes b,0) =$$
$$((a\otimes b)x\otimes a\otimes b,0).$$

(iii) F is an embedding: Let $a,b\in A$, $a\neq b$. Then $(ea\otimes a,0)\neq(eb\otimes b,0)$. As

$$F(a)((e,1)) = (ae\otimes a,0) = (a,0) \text{ and } F(b)((e,1)) = (be\otimes b,0) = (b,0)$$

we have that there exists an element $(e,1)$ of B such that $F(a)((e,1)) \neq F(b)((e,1))$. Hence, $F(a) \neq F(b)$.

(iv) F is injective: Let $a, b \in A$ such that $F(a) = F(b)$. This means $F(a)((x,i)) = F(b)((x,i))$ for arbitrary $x \in A^e$, $i \in \{0,1\}$. Specially, for $x = e$ and $i = 1$, we have

$$(a,0) = (ae \otimes a, 0) = F(a)((e,1)) = F(b)((e,1)) = (be \otimes b, 0) = (b,0).$$

Thus, $a = b$. and the mapping F is an injective. □

A semilattice-ordered semigroup $(A, \cdot, \otimes)$ is said to be representable with binary relations if there is a set B and an injective homomorphism $f\colon (A, \cdot, \otimes) \longrightarrow (Rel(B), \circ, \cup)$.

Let $F(S)$ be the set of all inhabited finite detachable subsets of S with the natural multiplication and join as in Example 3, then $(F(S), \cdot, \cup)$ is a semilattice-ordered semigroups. Let us note again that the relation $\neq_1$ on $F(S)$, defined as above, is not apartness.

The following result is a special case and we provide here a direct proof.

Theorem 2.2. *Let $(S, =, \neq, \cdot)$ be a semigroup with apartness. Then the semilattice-ordered semigroup $(\wp(S), \cdot, \cup)$ is representable with binary relations.*

Proof. Let $A \in \wp(S)$. Define a relation $f(A)$ on S^1 in the following way: For any $x, y \in S^1$,

$$(x,y) \in f(A) \Longleftrightarrow (\exists a \in A)(x = ay).$$

We will show that the mapping $f\colon P(S) \longrightarrow Rel(S^1)$ is a strongly extensional injective homomorphism from $(\wp(S), \cdot, \cup)$ to $(Rel(S^1), \circ, \cup)$.

Let $A, B \in \wp(S)$.

(a) $f(AB) = f(A) \circ f(B)$: Let $x, y \in S^1$, $xf(AB)y$. There is $c \in AB$ such that $x = cy$. But $c = ab$ for some $a \in A$, $b \in B$. Then $x = aby$. Put $z = by$. It holds: $xf(A)z$, $zf(B)y$. So, $x(f(A) \circ f(B))y$. Opposite, let $x, y \in S^1$ be elements such that, $x(f(A) \circ f(B))y$. There is $z \in S^1$ such that $xf(A)z$, $zf(B)y$. Then $x = az$, $z = by$ for some $a \in A$, $b \in B$. Now, $x = az = aby$ and $xf(AB)y$ since $ab \in AB$.

(b) $f(A \cup B) = f(A) \cup f(B)$: Let $x, y \in S^1$, $xf(A \cup B)y$. There is $c \in A \cup B$ such that $x = cy$. If $c \in A$ then $xf(A)y$. If $c \in B$ then $xf(B)y$. In any case, $x(f(A) \cup f(B))y$. Opposite, let $x, y \in S^1$, $x(f(A) \cup f(B))y$. If $xf(A)y$ then $x = ay$ for some $a \in A$. If $xf(B)y$ then $x = by$ for some $b \in B$. In any case $xf(A \cup B)y$ since $a, b \in A \cup B$.

(c) $A \neq B \Longrightarrow f(A) \neq f(B)$: Let $A \neq B$. Then $(\exists a \in A)(a \bowtie B)$ or $(\exists b \in B)(b \bowtie A)$. Let, for instance, $(\exists a \in A)(a \bowtie B)$. Clearly, $af(A)1$ and $(a,1) \neq (b,1) \in f(B)$ for any b of Y. We will show that $\neg(af(B)1)$. Suppose that $af(B)1$. Then there exists $b \in B$ with $a = b \cdot 1 = b \in B$ which is a contradiction.

(d) $f(A) = f(B)$ for some A and B of $\wp(S)$. Specially, from $(a,1) \in f(A)$ we have $(a,1) \in f(B)$ for any a of A, i.e. $a = b \cdot 1 = b \in B$. So, $A \subseteq B$. Analogously, we got $B \subseteq A$. Therefore, $A = B$.

(e) $f(A) \neq f(B) \Longrightarrow A \neq B$: Let $f(A) \neq f(B)$, that is, let $(\exists\, x, y \in S)((x,y) \in f(A) \wedge (x,y) \bowtie f(B))$ or $(\exists\, x, y \in S)((x,y) \in f(B) \wedge (x,y) \bowtie f(A))$. Let we analyze only the first case. The second case we can analyze analogously. So, there exists an element $a \in A$ such that $x = ay$ Clearly $\neg(A = B)$. Let (u,v) be arbitrary element of $f(B)$. Then $(x,y) \neq (u,v)$. Specially, we have $(x,y) \neq (by, y) \in f(B)$ for any b of B. Thus, $x \neq by$. Hence, $x \neq ax$ or $ax \neq by$ by cotransitivity of $\neq$. So, out of $ay \neq by$ we have $a \neq b$. According to selection of element b of B, we have $a \bowtie B$. It means $A \neq B$. □

Note that in both set $(\wp(S), \cdot, \cup)$ and $(Rel(S^1), \circ, \cup)$ diversity relation is not apartness.

Proposition 2.1. *The free semilattice-ordered semigroup $F(X^+)$ is representable by binary relations.*

Acknowledgment. The author thanks anonymous referees for their valuable suggestions for improvement of the paper.

REFERENCES

[1] H. Andreka: *Representations of distributive lattice-ordered semigroups with binary relations*, Algebra Universalis, **28**(1991), 12-25.
[2] G. Birkhoff: *Lattice Theory*, Providence, Rhode Island, 1967.
[3] E. Bishop: *Foundations of Constructive Analysis*, McGraw-Hill, New York, 1967.
[4] E. Bishop and D. S. Bridges: *Constructive Analysis*, Grundlehren der mathematischen Wissenschaften 279, Springer, Berlin, 1985.
[5] D. S. Bridges and F. Richman: *Varieties of Constructive Mathematics*, London Mathematical Society Lecture Notes 97, Cambridge University Press, Cambridge, 1987.
[6] D. S. Bridges and L. S. Vita: *Techniques of Constructive Analysis*, Springer, New York, 2006.
[7] D. van Dalen and F. J. de Vries: *Intuitionistic free abelian groups*, Math. Logic Quart., **34**(1988), No. 1, 3-12.
[8] R. Mines, F. Richman and W. Ruitenburg: *A Course of Constructive Algebra*, Springer-Verlag, New York, 1988.
[9] M. Mitrović, D. A. Romano and M. Vinčić: *A theorem on semilattice-ordered semigroup*, Int. Math. Forum, **4**(2009), No. 5, 227-232
[10] D. A. Romano: *Construction of free abelian groups*, Sarajevo J. Math., **4**(1988), No. 1, 151-158.
[11] D. A. Romano: *A note on quasi-antiorder in semigroup*, Novi Sad J. Math., **37**(2007), No. 1, 3-8.
[12] D. A. Romano: *An isomorphism theorem for anti-ordered sets*, Filomat, **22**(2008), No. 1, 145-160.
[13] D. A. Romano: *On semilattice-ordered semigroups. A constructive point of view*, Scientific Studies and Research. Series Mathematics and Informatics, **21**(2011), No. 2 (to appear).
[14] D. A. Romano: *On quasi-antiorder relation on semigroups*; Matematički Vesnik, **64**(2012), No. 3 (to appear).
[15] A. S. Troelstra and D. van Dalen: *Constructivism in Mathematics, An Introduction*, Volume II; North-Holland, Amsterdam 1988.

Banja Luka University
Faculty of Mechanical Engineering
71 *Vojvoda Stepa Stepanovic Street*
78000 *Banja Luka, Bosnia and Herzegovina*
E-mail address: bato49@hotmail.com