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COINCIDENCE POINT AND FIXED POINT THEOREMS IN ORDERED CONE METRIC SPACES

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Abstract. In this paper we establish some coincidence point and fixed point theorems in partially ordered cone metric spaces for mappings satisfying weak contractive type inequalities. The main results are illustrated with examples. There are several corollaries which are versions of weak contraction principle in ordered cone metric spaces.

1. INTRODUCTION

Cone metric space is a recently introduced generalization of metric space where every pair of elements is assigned to an element of a Banach space equipped with a cone. The cone induces a natural partial order in the Banach space. Cone metric spaces was defined by Huang et al in [24]. Fixed point studies were initiated in such spaces in the same work. After that, fixed point theory has experienced a rapid growth in cone metric spaces. A review of this development is given in [26]. References [16, 21, 30, 31, 32] are some more recent examples of these work.

Weak contraction principle is a generalization of Banach's contraction principle which was first given by Alber et al. in Hilbert spaces [2]. It was subsequently extended to metric spaces by Rhoades [38] and further generalized by several authors like Dutta and Choudhury [20], Popescu [36], Choudhury and Kundu [15] etc. Weak contraction principle was established in cone metric spaces by the present authors in [11]. In the sprit of weak contraction, several weak contractive inequalities have been used to obtain fixed point and coincidence point results in metric spaces. References [13, 18, 40] are some examples of these works.

Khan et al. [33] initiated the use of a control function in metric fixed point theory, which they called an Altering distance function. This function and its generalizations have been used in fixed and coincidence point problems in a large number of works; some of these works are in [9, 10, 39]. Recently, the present authors have introduced the concept of control function in cone metric spaces [12].

In recent times, fixed point theory has also developed rapidly in partially ordered metric spaces; that is, metric spaces endowed with a partial ordering. Some of these works are noted in [8, 14, 23, 34, 35, 37]. Some fixed point results have also been deduced in partially ordered cone metric spaces as for example, in [3] and [28].

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Compatibility between two mappings on a metric space is a concept which was defined by Jungck [27]. It is a condition which is weaker than commuting condition. Compatibility has been used in several common fixed point and coincidence point problems in metric spaces. Some works in this line of research are in [5, 6, 7]. Recently, in [4] the concept of compatibility between two mappings has been introduced in cone metric space.

In this paper we have proved some coincidence point and fixed point results using a control function in cone metric spaces having a partial order for pair of mappings where these mappings are assumed to satisfy certain weak inequalities. In one of our results, functions are assumed to be continuous and compatible. In the other result, the conditions of continuity and compatibility have been relaxed by taking a property of the cone metric space. Our results are supported with examples.

Despite [1, 19, 22, 29] it may so happen that an inequality in the cone metric space is not reducible to a metric inequality of the same form. In our case a weak contraction in a ordered cone metric space is not a weak contraction in the corresponding ordered metric space. In general, neither of the methods discussed in the above mentioned papers can convert a weak inequality in a cone metric space even to a corresponding metric inequality. Further, the relation of the partial ordering with the cone metric is not the same as its relation with the corresponding generated metric. It is doubtful whether the problem can be treated as a contractive fixed point problem in the metric space which obviously it is in a cone metric space.

In the following section we describe some definitions and existing results which we use in establishing our theorems.

2. PRELIMINARIES

Definition 2.1 ([24])**.** Let E always be a real Banach space and P a subset of E. P is called a *cone* if and only if:

i) P is nonempty, closed, and $P \neq \{0\}$;

ii) $a, b \in \mathbb{R},\ \ a, b \geq 0,\ \ x, y \in P \Longrightarrow ax + by \in P$;

iii) $x \in P$ and $-x \in P \Longrightarrow x = 0$.

Given a cone $P \subset E$, a partial ordering $\leq$ with respect to P is naturally defined by $x \leq y$ if and only if $y - x \in P$, for $x, y \in E$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$, where int$P$ denotes the interior of P.

The cone P is said to be normal if there exists a real number $K \geq 1$ such that for all $x, y \in E$, $0 \leq x \leq y \Rightarrow \|x\| \leq K\|y\|$. The least positive number K satisfying the above statement is called the normal constant of P.

The cone P is called regular if every increasing sequence which is bounded from above is convergent; that is, if $\{x_n\}$ is a sequence such that

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y,$$

for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \longrightarrow 0$ as $n \longrightarrow \infty$. Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

In the following we always suppose that E is a real Banach space with cone P in E with int $P \neq \emptyset$ and $\leq$ is the partial ordering with respect to P.

Definition 2.2 ([24])**.** Let X be a nonempty set. Let the mapping $d\colon X \times X \longrightarrow E$ satisfies:

i) $0 \leq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

ii) $d(x,y) = d(y,x)$, for all $x, y \in X$;
iii) $d(x,y) \leq d(x,z) + d(z,y)$, for all $x, y, z \in X$.
Then d is called a *cone metric* on X and (X,d) is called a *cone metric space*.

Definition 2.3 ([24]). Let (X,d) be a cone metric space, $\{x_n\}$ a sequence in X and $x \in X$.
i) If for every $c \in E$ with $0 \ll c$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be *convergent* and $\{x_n\}$ converges to x, x is the limit of $\{x_n\}$. We denote this by $\lim\limits_n x_n = x$ or $x_n \longrightarrow x$ as $n \longrightarrow \infty$.
ii) If for every $c \in E$ with $0 \ll c$ there exists $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a *Cauchy sequence* in X.
iii) If every Cauchy sequence in X is convergent in X, then X is called a *complete* cone metric space.
If P is a normal cone, then $\{x_n\}$ converges to x if and only if $d(x_n, x) \longrightarrow 0$ as $n \longrightarrow \infty$ and $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \longrightarrow 0$ as $n, m \longrightarrow \infty$, [24].

Definition 2.4 ([12]). Let $\psi\colon P \longrightarrow P$ be a function.
i) We say ψ is *strongly monotone increasing* if for $x, y \in P$, $x \leq y \iff \psi(x) \leq \psi(y)$.
ii) ψ is said to be *continuous* at $x_0 \in P$ if for any sequence $\{x_n\}$ in P, $x_n \longrightarrow x_0 \Longrightarrow \psi(x_n) \longrightarrow \psi(x_0)$.

The following is the definition of altering distance function in cone metric space.

Definition 2.5. A function $\psi\colon P \longrightarrow P$ is called an *altering distance* function if the following properties are satisfied:
i) ψ is strongly monotone increasing and continuous;
ii) $\psi(t) = 0$ if and only if $t = 0$.

Definition 2.6 ([27]). Let f and g be self-maps of a set X (i.e., $f, g\colon X \longrightarrow X$). If $w = fx = gx$, for some $x \in X$, then x is called a *coincidence point* of f and g, and w is called a *point of coincidence* of f and g.

Definition 2.7 ([28]). Let $(X, \preceq)$ be a partially ordered set and $f\colon X \longrightarrow X$ and $g\colon X \longrightarrow X$. The mapping f is is said to be g - *nondecreasing* if for all $x, y \in X$, $gx \preceq gy$ implies $fx \preceq fy$ and g - nonincreasing if for all $x, y \in X$, $gx \preceq gy$ implies $fx \succeq fy$.

Definition 2.8 ([4]). Let (X,d) be a cone metric space and $f, g\colon X \longrightarrow X$ are given self-mappings on X. The pair (f,g) is said to be *compatible* if $\lim\limits_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} gx_n = t$, for some $t \in X$.

Lemma 2.1. *Let E be a real Banach space with cone P in E. Then:*
i) *if $a \leq b$ and $b \ll c$, then $a \ll c$,* [25];
ii) *if $a \ll b$ and $b \ll c$, then $a \ll c$,* [25];
iii) *if $0 \leq x \leq y$ and $a \geq 0$, where a is real number, then $0 \leq ax \leq ay$,* [25];
iv) *if $0 \leq x_n \leq y_n$, for $n \in \mathbb{N}$ and $\lim\limits_n x_n = x$, $\lim\limits_n y_n = y$, then $0 \leq x \leq y$,* [25];
v) *P is normal if and only if $x_n \leq y_n \leq z_n$ and $\lim\limits_n x_n = \lim\limits_n z_n = x$ imply $\lim\limits_n y_n = x$,* [17].

Lemma 2.2 ([12]). *Let (X,d) be a cone metric space with regular cone P such that $d(x,y) \in \operatorname{int} P$, for $x, y \in X$ with $x \neq y$. Let $\phi : \operatorname{int} P \cup \{0\} \longrightarrow \operatorname{int} P \cup \{0\}$ be a function with the following properties:*

i) $\phi(t) = 0$ *if and only if* $t = 0$;
ii) $\phi(t) \ll t$, *for* $t \in \operatorname{int} P$;
iii) *either* $\phi(t) \leq d(x, y)$ *or* $d(x, y) \leq \phi(t)$, *for* $t \in \operatorname{int} P \cup \{0\}$ *and* $x, y \in X$.
Let $\{x_n\}$ *be a sequence in* X *for which* $\{d(x_n, x_{n+1})\}$ *is monotonic decreasing. Then* $\{d(x_n, x_{n+1})\}$ *is convergent to either* $r = 0$ *or* $r \in \operatorname{int} P$.

3. MAIN RESULTS

Lemma 3.1. *Let* (X, d) *be a cone metric space. Let* ϕ: $\operatorname{int} P \cup \{0\} \longrightarrow \operatorname{int} P \cup \{0\}$ *be a function such that:*
i) $\phi(t) = 0$ *if and only if* $t = 0$;
ii) $\phi(t) \ll t$, *for* $t \in \operatorname{int} P$.
Then a sequence $\{x_n\}$ *in* X *is a Cauchy sequence if and only if for every* $c \in E$ *with* $0 \ll c$ *there exists* $n_0 \in \mathbb{N}$ *such that* $d(x_n, x_m) \ll \phi(c)$, *for all* $n, m > n_0$.

Proof. Let $\{x_n\}$ be a sequence in X. Suppose that $\{x_n\}$ is a Cauchy sequence. Then for every $c \in E$ with $0 \ll c$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$, for all $n, m > n_0$. Let $c \in E$ with $0 \ll c$ be arbitrary. By condition i) of the lemma $\phi(c) \in \operatorname{int} P$; that is, $0 \ll \phi(c)$. Therefore, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll \phi(c)$, for all $n, m > n_0$.

Conversely suppose that for every $c \in E$ with $0 \ll c$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll \phi(c)$, for all $n, m > n_0$.

Since $c \in \operatorname{int} P$, by condition ii) of the lemma, we have $\phi(c) \ll c$.

Combining the above two inequalities by using the property ii) of lemma 2.1, we obtain

$$d(x_n, x_m) \ll c, \quad \text{for all } n, m > n_0.$$

Therefore, for every $c \in E$ with $0 \ll c$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$, for all $n, m > n_0$. Hence $\{x_n\}$ is a Cauchy sequence. □

Theorem 3.1. *Let* $(X, \preceq, d)$ *be an ordered cone metric space with regular cone* P *such that* $d(x, y) \in \operatorname{int} P$, *for* $x, y \in X$ *with* $x \neq y$. *Let* f *and* g *be continuous self mappings on* X *such that* $f(X) \subseteq g(X)$, f *is* g *- nondecrasing with respect to* $\preceq$ *and* (f, g) *is compatible pairs. Further, let for all* $x, y \in X$ *with* $gy \preceq gx$, f *and* g *satisfy the following inequality*

$$\psi(d(fx, fy)) + p\ d(gx, gy) \leq\ p\ u(x, y) + \psi(d(gx, gy)) - \phi(d(gx, gy)), \tag{3.1}$$

where, $u(x, y) \in \{d(gx, fy), d(gy, fy)\}$, $p \geq 0$ *a real number and* ψ: $P \longrightarrow P$, ϕ: $\operatorname{int} P \cup \{0\}$ $\longrightarrow \operatorname{int} P \cup \{0\}$ *are continuous functions with the following properties:*
i) ψ *is strongly monotonic increasing;*
ii) $\phi(t) = 0 = \psi(t)$ *if and only if* $t = 0$;
iii) $\phi(t) \ll t$, *for* $t \in intP$;
iv) *either* $\phi(t) \leq d(x, y)$ *or* $d(x, y) \ll \phi(t)$, *for* $t \in intP \cup \{0\}$ *and* $x, y \in X$.
If $g(X)$ *is a complete subspace of* X *and there exists* $x_0 \in X$ *such that* $gx_0 \preceq fx_0$, *then* f *and* g *have a coincidence point in* X.

Proof. Let $x_0 \in X$ be such that $gx_0 \preceq fx_0$. Since $f(X) \subseteq g(X)$, we can choose $x_1 \in X$ such that $gx_1 = fx_0$. Again we can choose $x_2 \in X$ such that $gx_2 = fx_1$. Continuing this process we construct a sequence $\{x_n\}$ in X such that

$$gx_{n+1} = fx_n, \quad \text{for all } n \geq 0. \tag{3.2}$$

Since $gx_0 \preceq fx_0$ and $gx_1 = fx_0$, we have $gx_0 \preceq gx_1$ which implies that $fx_0 \preceq fx_1$. Now, $fx_0 \preceq fx_1$, that is, $gx_1 \preceq gx_2$ implies that $fx_1 \preceq fx_2$. Again, $fx_1 \preceq fx_2$, that is, $gx_2 \preceq gx_3$ implies that $fx_2 \preceq fx_3$. Continuing this process, we have

$$gx_0 \preceq gx_1 \preceq gx_2 \preceq gx_3 \preceq \cdots \preceq gx_n \preceq gx_{n+1} \preceq \cdots,$$

and

$$fx_0 \preceq fx_1 \preceq fx_2 \preceq fx_3 \preceq \cdots \preceq fx_n \preceq fx_{n+1} \preceq \cdots$$

Since for $x = x_n$ and $y = x_{n-1}$, $gx_{n-1} \preceq gx_n$, from (3.1), we have

$$\begin{aligned}\psi(d(gx_{n+1}, gx_n)) + p\ d(gx_n, gx_{n-1}) &= \psi(d(fx_n, fx_{n-1})) + p\ d(gx_n, gx_{n-1}) \\ &\leq p\ u(x_n, x_{n-1}) + \psi(d(gx_n, gx_{n-1})) - \phi(d(gx_n, gx_{n-1})), \quad (3.3)\end{aligned}$$

where $u(x_n, x_{n-1}) \in \{d(gx_n, fx_{n-1}), d(gx_{n-1}, fx_{n-1})\} = \{d(gx_n, gx_n), d(gx_{n-1}, gx_n)\}$, that is, $u(x_n, x_{n-1}) \in \{0, d(gx_{n-1}, gx_n)\}$.

If $u(x_n, x_{n-1}) = 0$, then from (3.3), we have

$$\psi(d(gx_{n+1}, gx_n)) + p\ d(gx_n, gx_{n-1}) \leq \psi(d(gx_n, gx_{n-1})) - \phi(d(gx_n, gx_{n-1})),$$

which implies

$$\psi(d(gx_{n+1}, gx_n)) \leq \psi(d(gx_n, gx_{n-1})) - \phi(d(gx_n, gx_{n-1})).$$

If $u(x_n, x_{n-1}) = d(gx_{n-1}, gx_n)$, then from (3.3), we have

$$\psi(d(gx_{n+1}, gx_n)) + p\ d(gx_n, gx_{n-1}) \leq p\ d(gx_{n-1}, gx_n) + \psi(d(gx_n, gx_{n-1})) - \phi(d(gx_n, gx_{n-1})),$$

that is,

$$\psi(d(gx_{n+1}, gx_n)) \leq \psi(d(gx_n, gx_{n-1})) - \phi(d(gx_n, gx_{n-1})).$$

Therefore, from (3.3), we have

$$\psi(d(gx_{n+1}, gx_n)) \leq \psi(d(gx_n, gx_{n-1})) - \phi(d(gx_n, gx_{n-1})). \quad (3.4)$$

Using a property of ϕ, we have for all $n \geq 1$,

$$\psi(d(gx_{n+1}, gx_n)) \leq \psi(d(gx_n, gx_{n-1})),$$

which implies that for all $n \geq 1$, $d(gx_{n+1}, gx_n) \leq d(gx_n, gx_{n-1})$ (by a property of ψ).

Therefore, $\{d(gx_{n+1}, gx_n)\}$ is a monotone decreasing sequence. Hence by lemma 2.2, there exists an $r \in P$ with either $r = 0$ or $r \in \text{int } P$ such that

$$d(gx_{n+1}, gx_n) \longrightarrow r \text{ as } n \longrightarrow \infty. \quad (3.5)$$

Taking the limit as $n \to \infty$ in (3.4) and using (3.5) and continuities of ϕ and ψ, we have

$$\psi(r) \leq \psi(r) - \phi(r),$$

which is a contradiction unless $r = 0$. Therefore,

$$d(gx_{n+1}, gx_n) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \quad (3.6)$$

Next we show that $\{gx_n\}$ is a Cauchy sequence. If $\{gx_n\}$ is not a Cauchy sequence, then by lemma 3.1, there exists a $c \in E$ with $0 \ll c$ such that $\forall\ n_0 \in \mathbb{N}$, $\exists\ n, m \in \mathbb{N}$ with $n > m > n_0$ such that $d(gx_n, gx_m) <\not\ll \phi(c)$. Hence by a property of ϕ in (iv) of the theorem, $\phi(c) \leq d(gx_n, gx_m)$.

Therefore, there exist sequences $\{n(k)\}$ and $\{m(k)\}$ in $\mathbb{N}$ such that for all positive integers k, $n(k) > m(k) > k$ and $d(gx_{n(k)}, gx_{m(k)}) \geq \phi(c)$. Assuming that $n(k)$ is the

smallest such positive integer, we get $d(gx_{n(k)}, gx_{m(k)}) \geq \phi(c)$ and $d(gx_{n(k)-1}, gx_{m(k)}) \ll \phi(c)$. Now, $\phi(c) \leq d(gx_{n(k)}, gx_{m(k)}) \leq d(gx_{n(k)}, gx_{n(k)-1}) + d(gx_{n(k)-1}, gx_{m(k)})$, that is,

$$\phi(c) \leq d(gx_{n(k)}, gx_{m(k)}) \leq d(gx_{n(k)}, gx_{n(k)-1}) + \phi(c).$$

Letting $k \longrightarrow \infty$ in the above inequality, using (3.6) and the property v) of Lemma 2.1, we have

$$\lim_{k\to\infty} d(gx_{n(k)}, gx_{m(k)}) = \phi(c). \tag{3.7}$$

Again,

$$d(gx_{n(k)}, gx_{m(k)}) \leq d(gx_{n(k)}, gx_{n(k)+1}) + d(gx_{n(k)+1}, gx_{m(k)+1}) + d(gx_{m(k)+1}, gx_{m(k)})$$

and

$$d(gx_{n(k)+1}, gx_{m(k)+1}) \leq d(gx_{n(k)+1}, gx_{n(k)}) + d(gx_{n(k)}, gx_{m(k)}) + d(gx_{m(k)}, gx_{m(k)+1}).$$

Letting $k \longrightarrow \infty$ in above inequalities, using (3.6) and (3.7), we have

$$\lim_{k\to\infty} d(gx_{n(k)+1}, gx_{m(k)+1}) = \phi(c). \tag{3.8}$$

Again,

$$d(gx_{n(k)}, gx_{m(k)}) \leq d(gx_{n(k)}, gx_{m(k)+1}) + d(gx_{m(k)+1}, gx_{m(k)})$$

and

$$d(gx_{n(k)}, gx_{m(k)+1}) \leq d(gx_{n(k)}, gx_{m(k)}) + d(gx_{m(k)}, gx_{m(k)+1}).$$

Letting $k \longrightarrow \infty$ in above inequalities, using (3.6) and (3.7), we have

$$\lim_{k\to\infty} d(gx_{n(k)}, gx_{m(k)+1}) = \phi(c). \tag{3.9}$$

Since for $x = x_{n(k)}$ and $y = x_{m(k)}$, $gx_{m(k)} \preceq gx_{n(k)}$, from (3.1), we have

$$\psi(d(gx_{n(k)+1}, gx_{m(k)+1})) + p\ d(gx_{n(k)}, gx_{m(k)}) \leq p\ u(x_{n(k)}, x_{m(k)}) + \psi(d(gx_{n(k)}, gx_{m(k)})) - \phi(d(gx_{n(k)}, gx_{m(k)})),$$

where $u(x_{n(k)}, x_{m(k)}) \in \{d(gx_{n(k)}, gx_{m(k)+1}), d(gx_{m(k)}, gx_{m(k)+1})\}$.

Letting $k \longrightarrow \infty$ in the above inequality, using (3.6), (3.7), (3.8) and (3.9) and using the continuities of ϕ and ψ, we have

$$\psi(\phi(c)) + p\ \phi(c) \leq p\ \phi(c) + \psi(\phi(c)) - \phi(\phi(c)),$$

or,

$$\psi(\phi(c)) + p\ \phi(c) \leq \psi(\phi(c)) - \phi(\phi(c)),$$

according as $u(x_{n(k)}, x_{m(k)}) = d(gx_{n(k)}, gx_{m(k)+1})$ or $u(x_{n(k)}, x_{m(k)}) = d(gx_{m(k)}, gx_{m(k)+1})$ respectively. It follows that

$$\psi(\phi(c)) \leq \psi(\phi(c)) - \phi(\phi(c)),$$

which is a contradiction by virtue of a property of ϕ. Hence $\{gx_n\}$ is a Cauchy sequence. From the completeness of $g(X)$, there exists $z \in X$ such that

$$gx_n \longrightarrow gz \ \text{ as } \ n \longrightarrow \infty. \tag{3.10}$$

Again,

$$fx_n = gx_{n+1} \longrightarrow gz \ \text{ as } \ n \longrightarrow \infty. \tag{3.11}$$

Since f and g are continuous and compatible, and $gx_n \longrightarrow gz$, $fx_n \longrightarrow gz$ as $n \longrightarrow \infty$,

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0.$$

It follows that $d(fgz, ggz) = 0$, that is, $fgz = ggz$, that is, $gz \in X$ is a coincidence point of f and g. □

Theorem 3.2. *Let $(X, \preceq, d)$ be an ordered cone metric space with regular cone P such that $d(x, y) \in intP$, for $x, y \in X$ with $x \neq y$. Assume that if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \longrightarrow x$ then $x_n \preceq x$, for all $n \in \mathbb{N}$. Let f and g be self mappings on X such that $f(X) \subseteq g(X)$, f is g - nondecrasing with respect to $\preceq$. Suppose that* (3.1) *holds, and the conditions upon p and (ϕ, ψ) are the same as in Theorem* 3.1. *If $g(X)$ is a complete subspace of X and there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$, then f and g have a coincidence point in X.*

Proof. We take the same sequence $\{x_n\}$ as in the proof of Theorem 3.1. Then we have $gx_0 \preceq gx_1 \preceq gx_2 \preceq gx_3 \preceq \cdots \preceq gx_n \preceq gx_{n+1} \preceq \cdots$, and $fx_0 \preceq fx_1 \preceq fx_2 \preceq fx_3 \preceq \cdots \preceq fx_n \preceq fx_{n+1} \preceq \cdots$; that is, $\{gx_n\}$ and $\{fx_n\}$ are nondecreasing sequences. Also, this sequences converge to gz. Then $gx_n \preceq gz$, for all $n \in \mathbb{N}$. Therefore, we can use the condition (3.1) for $x = z$, $y = x_n$ and so we have

$$\psi(d(fz, fx_n)) + p\ d(gz, gx_n) \leq p\ u(z, x_n) + \psi(d(gz, gx_n)) - \phi(d(gz, gx_n)),$$

where $u(z, x_n) \in \{d(gz, fx_n), d(gx_n, fx_n)\}$.

Taking the limit as $n \longrightarrow \infty$ in the above inequality, using the properties of ϕ and ψ, we have $\psi(d(fz, gz)) \leq 0$. It follows by a property of ψ that $d(fz, gz) = 0$, that is, $fz = gz$, that is, z is a coincidence point of f and g. □

Considering g to be the identity function in theorems 3.1 and 3.2, we have following corollaries.

Corollary 3.1. *Let $(X, \preceq, d)$ be a complete ordered cone metric space with regular cone P such that $d(x, y) \in \operatorname{int} P$, for $x, y \in X$ with $x \neq y$. Let $f\colon X \longrightarrow X$ be a continuous and nondecrasing mapping with respect to $\preceq$ such that for all $x, y \in X$ with $y \preceq x$,*

$$\psi(d(fx, fy)) + p\ d(x, y) \leq\ p\ u(x, y) + \psi(d(x, y)) - \phi(d(x, y)), \tag{3.12}$$

where $u(x, y) \in \{d(x, fy), d(y, fy)\}$, and the conditions upon p and (ϕ, ψ) are the same as in Theorem 3.1. *If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point in X.*

Corollary 3.2. *Let $(X, \preceq, d)$ be a complete ordered cone metric space with regular cone P such that $d(x, y) \in \operatorname{int} P$, for $x, y \in X$ with $x \neq y$. Assume that if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \longrightarrow x$ then $x_n \preceq x$, for all $n \in \mathbb{N}$. Let $f\colon X \longrightarrow X$ be a nondecrasing mapping with respect to $\preceq$. Suppose that* (3.12) *holds, and the conditions upon p and (ϕ, ψ) are the same as in Theorem* 3.1. *If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point in X.*

Theorem 3.3. *In addition to the hypotheses of Corollary* 3.1 *and Corollary* 3.2, *in both of the corollaries, suppose that for every $x, y \in X$ there exists a $z \in X$ such that $x \preceq z$ and $y \preceq z$. Then f has a unique fixed point.*

Proof. It follows from the corollary 3.1 or corollary 3.2, the set of fixed points of f is non-empty. If possible let $x, y \in X$ $(x \neq y)$ be two fixed points of f. We distinguish two cases:

Case 1. If $y \preceq x$, then $f^n y = y \preceq x = f^n x$, for $n = 0, 1, 2, \ldots$. By the condition (3.12), we have for all $n \geq 1$,

$$\begin{aligned}\psi(d(x,y)) + p\ d(x,y) &= \psi(d(f^n x, f^n y)) + p\ d(f^{n-1}x, f^{n-1}y) \\ &\leq\ p\ u(f^{n-1}x, f^{n-1}y) + \psi(d(f^{n-1}x, f^{n-1}y)) - \phi(d(f^{n-1}x, f^{n-1}y)),\end{aligned}$$

that is,

$$\psi(d(x,y)) + p\ d(x,y) \leq\ p\ u(x,y) + \psi(d(x,y)) - \phi(d(x,y)),$$

that is,

$$p\ d(x,y) \leq\ p\ u(x,y) - \phi(d(x,y)),$$

where $u(x,y) \in \{d(x,y), 0\}$.

For $u(x,y) = d(x,y)$ or $u(x,y) = 0$, by a property of ϕ, it follows from the above inequality that $d(x,y) = 0$; that is, $x = y$.

Case 2. If $y \npreceq x$, then there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$. Monotonicity of f implies that $f^n x = x \preceq f^n z$ and $f^n y = y \preceq f^n z$, for $n = 0, 1, 2, \ldots$. By the condition (3.12), we have for all $n \geq 1$,

$$\begin{aligned}\psi(d(f^n z, x)) + p\ d(f^{n-1}z, x) &= \psi(d(f^n z, f^n x)) + p\ d(f^{n-1}z, f^{n-1}x) \\ &\leq\ p\ u(f^{n-1}z, f^{n-1}x) + \psi(d(f^{n-1}z, f^{n-1}x)) - \phi(d(f^{n-1}z, f^{n-1}x)),\end{aligned}$$

that is,

$$\psi(d(f^n z, x)) + p\ d(f^{n-1}z, x) \leq\ p\ u(f^{n-1}z, x) + \psi(d(f^{n-1}z, x)) - \phi(d(f^{n-1}z, x)),$$

where $u(f^{n-1}z, x) \in \{d(f^{n-1}z, x), d(x,x)\} = \{d(f^{n-1}z, x), 0\}$.

For $u(f^{n-1}z, x) = d(f^{n-1}z, x)$ or $u(f^{n-1}z, x) = 0$, it follows from the above inequality

$$\psi(d(f^n z, x)) \leq\ \psi(d(f^{n-1}z, x)) - \phi(d(f^{n-1}z, x)). \tag{3.13}$$

Using a property of ϕ, we have

$$\psi(d(f^n z, x)) \leq\ \psi(d(f^{n-1}z, x)),$$

which implies

$$d(f^n z, x) \leq\ d(f^{n-1}z, x), \quad \text{(by monotone property of } \psi).$$

Therefore, $\{d(f^n z, x)\}$ is a monotone decreasing sequence. Following the lemma 2.2, it can be proved that there exists a $q \in \text{int}P \cup \{0\}$ such that

$$\lim_{n\to\infty} d(f^n z, x) = q. \tag{3.14}$$

Letting $n \longrightarrow \infty$ in (3.13), using (3.14) and the continuities of ϕ and ψ, we have

$$\psi(q) \leq \psi(q) - \phi(q),$$

which is a contradiction unless $q = 0$.

Hence

$$\lim_{n\to\infty} d(f^n z, x) = 0.$$

Analogously, it can proved that

$$\lim_{n\to\infty} d(f^n z, y) = 0.$$

Finally, the uniqueness of the limit gives us $x = y$.

From above two cases we have that fixed point of f is unique. Hence the proof is completed. □

Taking $p = 0$ in theorems 3.1 and 3.2, we have the following corollaries.

Corollary 3.3. *Let $(X, \preceq, d)$ be an ordered cone metric space with regular cone P such that $d(x, y) \in \operatorname{int} P$, for $x, y \in X$ with $x \neq y$. Let f and g be continuous self mappings on X such that $f(X) \subseteq g(X)$, f is g - nondecrasing with respect to $\preceq$ and (f, g) is compatible pairs. Further, let for all $x, y \in X$ with $gy \preceq gx$, f and g satisfy the following inequality*

$$\psi(d(fx, fy)) \leq \psi(d(gx, gy)) - \phi(d(gx, gy)), \tag{3.15}$$

where, the conditions upon (ϕ, ψ) are the same as in Theorem 3.1. *If $g(X)$ is a complete subspace of X and there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$, then f and g have a coincidence point in X.*

Corollary 3.4. *Let $(X, \preceq, d)$ be an ordered cone metric space with regular cone P such that $d(x, y) \in \operatorname{int} P$, for $x, y \in X$ with $x \neq y$. Assume that if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \longrightarrow x$ then $x_n \preceq x$, for all $n \in \mathbb{N}$. Let f and g be self mappings on X such that $f(X) \subseteq g(X)$, f is g - nondecrasing with respect to $\preceq$. Suppose that* (3.15) *holds, and the conditions upon (ϕ, ψ) are the same as in Theorem* 3.1. *If $g(X)$ is a complete subspace of X and there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$, then f and g have a coincidence point in X.*

Remark 3.1. Corollaries 3.3 and 3.4 are versions of theorem 3.1 of [12] in ordered cone metric spaces.

Taking $p = 0$ and g to be the identity mapping in theorems 3.1 and 3.2, we have the following corollaries.

Corollary 3.5. *Let $(X, \preceq, d)$ be a complete ordered cone metric space with regular cone P such that $d(x, y) \in \operatorname{int} P$, for $x, y \in X$ with $x \neq y$. Let $f \colon X \longrightarrow X$ be a continuous and nondecrasing mapping with respect to $\preceq$ such that for all $x, y \in X$ with $y \preceq x$,*

$$\psi(d(fx, fy)) \leq \psi(d(x, y)) - \phi(d(x, y)), \tag{3.16}$$

where the conditions upon (ϕ, ψ) are the same as in Theorem 3.1. *If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point in X.*

Corollary 3.6. *Let $(X, \preceq, d)$ be a complete ordered cone metric space with regular cone P such that $d(x, y) \in intP$, for $x, y \in X$ with $x \neq y$. Assume that if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \longrightarrow x$ then $x_n \preceq x$, for all $n \in \mathbb{N}$. Let $f \colon X \longrightarrow X$ be a nondecrasing mapping with respect to $\preceq$. Suppose that* (3.16) *holds, and the conditions upon (ϕ, ψ) are the same as in Theorem* 3.1. *If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point in X.*

Taking $p = 0$ and g and ψ to be the identity mappings in theorems 3.1 and 3.2, we have the following corollaries.

Corollary 3.7. *Let $(X, \preceq, d)$ be a complete ordered cone metric space with regular cone P such that $d(x, y) \in \operatorname{int} P$, for $x, y \in X$ with $x \neq y$. Let $f \colon X \longrightarrow X$ be a continuous and nondecrasing mapping with respect to $\preceq$ such that for all $x, y \in X$ with $y \preceq x$,*

$$d(fx, fy) \leq d(x, y) - \phi(d(x, y)), \tag{3.17}$$

where the conditions upon ϕ are the same as in Theorem 3.1. *If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point in X.*

Corollary 3.8. *Let $(X, \preceq, d)$ be a complete ordered cone metric space with regular cone P such that $d(x, y) \in \operatorname{int} P$, for $x, y \in X$ with $x \neq y$. Assume that if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \longrightarrow x$ then $x_n \preceq x$, for all $n \in \mathbb{N}$. Let $f\colon X \longrightarrow X$ be a nondecrasing mapping with respect to $\preceq$. Suppose that* (3.17) *holds, and the conditions upon ϕ are the same as in Theorem* 3.1*. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point in X.*

Remark 3.2. Corollaries 3.7 and 3.8 are versions of theorem 2.1 of [11] in ordered cone metric spaces.

Remark 3.3. In corollaries 3.5, 3.6, 3.7 and 3.8, the function f will have unique fixed point if in each of these corollaries we take the additional condition that for every $x, y \in X$ there exists a $z \in X$ which is comparable to both x and y.

Example 3.1. Let $X = [0, 1]$ with usual order $\preceq$ be a partially ordered set. Let $E = \mathbb{R}^2$, with usual norm, be a real Banach space. We define $P = \{(x, y) \in E : x, y \geq 0\}$. The partial ordering $\leq$ with respect to the cone P be the partial ordering in E. Then P is a regular cone. Let $d\colon X \times X \longrightarrow E$ be given as

$$d(x, y) = (\mid x - y \mid, \mid x - y \mid), \text{ for } x, y \in X.$$

Then $(X, \preceq, d)$ is an ordered cone metric space with the required properties of Theorem 3.1 and 3.2.

Let $\psi\colon P \longrightarrow P$ and $\phi\colon \operatorname{int} P \cup \{0\} \longrightarrow \operatorname{int} P \cup \{0\}$ be defined respectively as follows:

For $t = (x, y) \in P$,

$$\psi(t) = \begin{cases} (x, y), & x \leq 1,\ y \leq 1 \\ (x^2, y), & x > 1,\ y \leq 1 \\ (x, y^2), & x \leq 1,\ y > 1 \\ (x^2, y^2), & x > 1,\ y > 1, \end{cases}$$

and for $s = (s_1, s_2) \in \operatorname{int} P \cup \{0\}$ with $v = \min\{s_1, s_2\}$,

$$\phi(s) = \begin{cases} \left(\dfrac{v^2}{2}, \dfrac{v^2}{2}\right), & v \leq 1 \\ \left(\dfrac{1}{2}, \dfrac{1}{2}\right), & v > 1. \end{cases}$$

Then ψ and ϕ have the properties mentioned in Theorems 3.1 and 3.2.

Let $p = \dfrac{1}{4}$.

A) Let $f, g\colon X \longrightarrow X$ be defined respectively as follows:

$$fx = \frac{x}{32} \text{ and } g(x) = \frac{x}{4}, \text{ for } x \in X.$$

Then f and g have the required properties mentioned in Theorem 3.1.

It can be verified that for all $x, y \in X$ with $gy \preceq gx$, (3.1) is satisfied. Hence the conditions of Theorem 3.1 are satisfied and it is seen that 0 is a coincidence points of f and g.

B) Let $f, g\colon X \longrightarrow X$ be defined respectively as follows:

$$fx = \begin{cases} 0, & 0 \leq x \leq \dfrac{1}{2} \\ \dfrac{1}{32}, & \dfrac{1}{2} < x \leq 1, \end{cases}$$

and

$$g(x) = \frac{x}{4}, \quad \text{for} \quad x \in X.$$

Then f and g have the required properties mentioned in Theorem 3.2.

It can be verified that for all for all $x, y \in X$ with $gy \preceq gx$, (3.1) is satisfied. Hence the conditions of Theorem 3.2 are satisfied and it is seen that 0 is a coincidence points of f and g.

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