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A DECISION FUNCTIONAL FOR MULTITIME CONTROLLABILITY

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Abstract. This paper refers to multitime linear normal PDE systems from controllability point of view. The ideas presented here are complementary to those in our most recent papers and contain original results on multitime controllability, based on the γ-gramian matrix, the Im-gramian space and a controllability functional. Some original examples emphasize the utility and novelty of the theoretical results having in mind possible applications to control and signal processing, circuits, image processing, computer tomography, seismology etc.

1. INTRODUCTION

Let $\mathbb{R}^m$ be endowed with the *product order*. The noun "multitime" means a multidimensional (mD) parameter $t = (t^1, \dots, t^m) \in \mathbb{R}^m$ of evolution $x(t) = (x^1(t), \dots, x^n(t)) \in \mathbb{R}^n$ (see, submanifold theory), from the point $x(t_0)$ to the point $x(t)$, as "the cursor on the desktop". The components $t^\alpha, \alpha = 1, \dots, m$, of the multitime are either independent "times" (temporal variables) or independent "times" combined with spatial variables, or only spatial variables. The adjective "multitime (multitemporal)" is used to suggest that a multidimensional (mD) evolution $x(t) \in \mathbb{R}^n$ is in fact a deformation as in the elasticity theory. This point of view is opposite to the classical agreement that the temporal parameter (variable) is unidimensional, i.e., $t \in \mathbb{R}$ (single-time, and that only him is an evolution parameter), while the spatial parameter $x \in \mathbb{R}^n$ can be multidimensional (but this is not an evolution parameter). A single-time (1D) evolution means a curve $x(t) \in \mathbb{R}^n$, from the point $x(t_0)$ to the point $x(t)$. But, why the "time" would not be multidimensional as the actual physical and philosophical theories stipulate? The first who perceived the foregoing difference and have introduced the "multitime" concept was Dirac. To him joined those who understood that there exist phenomena and problems in which we cannot distinguish between the evolution variables and the static variables.

This article studies the controllability of multitime (mD) linear PDE systems using ingredients related to the multitime optimal control theory and, in particular, to the multitime maximum principle (see [4], [5], [10] - [14], [15]). Our results in this direction are complementary to those contained in the papers [8], [9], [1] - [3], [6], [7], [16] and are based on some original elements, such as: (i) the *Im-gramian space*; it generalizes the controllability gramian image defined in [3] and [8] for additional hypotheses (relations (2.5) in this paper); (ii) the *controllability γ-functional* and the *controllability functional*; they are the

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result of mixing our ideas with those in [16], where a similar, though single-time functional was analyzed.

The basic original results of this paper are contained in Theorems 4.1, 5.1 and Theorem 6.1. The first one gives necessary conditions for multitime controllability, via the *Im - gramian space*. Theorem 5.1 derives necessary conditions for multitime controllability, using the *controllability γ - functional*. Finally, the third theorem contains necessary and sufficient conditions for multitime controllability, expressed by means of *controllability functional*.

2. PRELIMINARY RESULTS

2.1. **Controllability of multitime linear PDE systems.** Let $D \subseteq \mathbb{R}^m$ be an open and convex subset. We consider the evolution PDE system

$$\frac{\partial x}{\partial t^\alpha} = M_\alpha(t)x + N_\alpha(t)u_\alpha(t), \quad \forall \alpha = \overline{1,m}, \tag{2.1}$$

where $t = (t^1, \dots, t^m) \in \mathbb{R}^m$ denotes the *multitime* or the evolution parameter, $x = (x^1, \dots, x^n)^\top \colon D \to \mathbb{R}^n = \mathcal{M}_{n,1}(\mathbb{R})$ is the state variable, $M_\alpha \colon D \to \mathcal{M}_n(\mathbb{R})$ are $\mathcal{C}^1$ quadratic matrix functions, $N_\alpha \colon D \to \mathcal{M}_{n,k}(\mathbb{R})$ are $\mathcal{C}^1$ rectangular matrix functions and $u_\alpha : D \to \mathbb{R}^k = \mathcal{M}_{k,1}(\mathbb{R})$ are $\mathcal{C}^1$ *control vectorial functions*, $\alpha = \overline{1,m}$.

The PDE system (2.1) is called *completely integrable* if $\forall (t_0, x_0) \in D \times \mathbb{R}^n$, there exists an open set $D_0 \subseteq \mathbb{R}^m$, with $t_0 \in D_0$ and $\exists x : D_0 \to \mathbb{R}^n$, $x(\cdot)$ differentiable, such that $x(\cdot)$ satisfies (2.1) on D_0 and $x(t_0) = x_0$. If so, the vectorial function $x(\cdot)$ shall be called *solution* for the Cauchy problem $\{(2.1),\, x(t_0) = x_0\}$.

The system (2.1) is completely integrable if and only if the following relations hold

$$\frac{\partial M_\alpha}{\partial t^\beta} + M_\alpha(t)M_\beta(t) = \frac{\partial M_\beta}{\partial t^\alpha} + M_\beta(t)M_\alpha(t), \tag{2.2}$$

$$\begin{aligned} &M_\alpha(t)N_\beta(t)u_\beta(t) + \frac{\partial N_\alpha}{\partial t^\beta}u_\alpha(t) + N_\alpha(t)\frac{\partial u_\alpha}{\partial t^\beta} \\ &\quad = M_\beta(t)N_\alpha(t)u_\alpha(t) + \frac{\partial N_\beta}{\partial t^\alpha}u_\beta(t) + N_\beta(t)\frac{\partial u_\beta}{\partial t^\alpha}, \end{aligned} \tag{2.3}$$

$\forall t \in D,\ \forall \alpha, \beta = \overline{1,m}$. It follows that each solution $x(\cdot)$ is a $\mathcal{C}^2$ - vectorial function and may be uniquely extended to a global solution ($x \colon D \to \mathbb{R}^n$) and, if two solutions coincide at a point, then they coincide on D. Hence, is natural, hereafter, to refer only to global solutions.

We recall that papers [3], [8] proved that relations (2.2) and (2.3) lead to the solution

$$x(t) = \chi(t, t_0)x_0 + \int\limits_{\gamma_{t_0,t}} \chi(t,s)N_\alpha(s)u_\alpha(s)\mathrm{d}s^\alpha, \quad \forall t \in D, \tag{2.4}$$

for the Cauchy problem $\{(2.1),\, x(t_0) = x_0\}$, where $\gamma_{t_0,t}$ is a piecewise $\mathcal{C}^1$ curve on D, going from the multitime t_0 toward the multitime t and $\chi(t,s)$ is the fundamental matrix associated to the PDE system, i.e., the matrix solution of the Cauchy problem (see [8])

$$\frac{\partial \chi}{\partial t^\alpha}(t,s) = M_\alpha(t)\chi(t,s), \quad \forall \alpha = \overline{1,m},$$

$$\chi(s,s) = I_n.$$

Definition 2.1. Suppose that the matrix functions $M_\alpha(\cdot)$ satisfy the relations (2.2), $\forall t \in D,\ \forall \alpha, \beta = \overline{1,m}$. The vector space

$$\mathcal{U} = \Big\{ u = (u_\alpha)_{\alpha=\overline{1,m}} \ \Big|\ u_\alpha \colon D \to \mathbb{R}^k = \mathcal{M}_{k,1}(\mathbb{R}), \text{of class } \mathcal{C}^1, \forall \alpha = \overline{1,m}, \text{ satisfying (2.3) for all } \alpha, \beta \Big\}$$

is called *the control space*, associated to the system (2.1). If $u \in \mathcal{U}$, we say that u is a control.

In conclusion, if the matrices $M_\alpha(\cdot)$ satisfy the relations (2.2), $\forall t \in D$, $\forall \alpha, \beta = \overline{1,m}$, then the system (2.1) is completely integrable if and only if $(u_\alpha)_{\alpha=\overline{1,m}}$ is a control function.

Definition 2.2. Let us consider the PDE system (2.1), with $M_\alpha(\cdot)$ satisfying (2.2).

a) The phase (t, x) is called *controllable* if there exists a point $s \in D$, with $s^\alpha > t^\alpha$, $\forall \alpha$, and there exists a control $u(\cdot)$ taking the phase (t, x) toward the phase $(s, 0)$.

b) Let $t_0, t \in D$, with $t_0^\alpha < t^\alpha$, $\forall \alpha$. The PDE system (2.1) is called *completely controllable* from t_0 to t if, for each point $x \in \mathbb{R}^n$, the phase (t_0, x) transfers into the phase $(t, 0)$, i.e., for each point x, the phase (t_0, x) is controllable with the same t.

Let us consider the PDE system (2.1), for which $M_\alpha(\cdot)$ verify relations (2.2). Taking $t_0, t \in D$, we consider the set

$$\mathcal{V}(t_0, t) := \left\{ \int\limits_{\gamma_{t_0,t}} \chi(t_0, s) N_\alpha(s) u_\alpha(s) \mathrm{d}s^\alpha \ \Big|\ (u_\alpha)_{\alpha=\overline{1,m}} \text{ is a control} \right\}.$$

If $(u_\alpha)_{\alpha=\overline{1,m}}$ is a control, then the curvilinear integral $\displaystyle\int\limits_{\gamma_{t_0,t}} \chi(t_0, s) N_\alpha(s) u_\alpha(s) \mathrm{d}s^\alpha$ is path independent, so $\mathcal{V}(t_0, t)$ depends not on the curve $\gamma_{t_0,t}$, joining t_0 and t, but only on the endpoints t_0 and t.

One remarks immediately that the set $\mathcal{V}(t_0, t)$ is a vector subspace of $\mathbb{R}^n$.

Definition 2.3. The space $\mathcal{V}(t_0, t)$ is called *the controllability space.*

Theorem 2.1. *Let us consider the PDE system* (2.1)*, with the matrix functions* $M_\alpha(\cdot)$ *satisfying* (2.2)*.*

i) *The control* $(u_\alpha)_{\alpha=\overline{1,m}}$ *transfers the phase* (t_0, x_0) *into the phase* (t, y) *if and only if* $\chi(t_0, t)y - x_0 = \displaystyle\int\limits_{\gamma_{t_0,t}} \chi(t_0, s) N_\alpha(s) u_\alpha(s) ds^\alpha$.

ii) *The phase* (t_0, x_0) *transfers into the phase* (t, y) *if and only if*

$$x_0 - \chi(t_0, t)y \in \mathcal{V}(t_0, t).$$

iii) *The phase* (t_0, x_0) *is controllable if and only if* $\exists t \in D$*, with* $t^\alpha > t_0^\alpha$*,* $\forall \alpha$ *such that* $x_0 \in \mathcal{V}(t_0, t)$*.*

iv) *Let* $t_0, t \in D$*, with* $t_0^\alpha < t^\alpha$*,* $\forall \alpha$*. The PDE system is completely controllable from* t_0 *to* t *if and only if* $\mathcal{V}(t_0, t) = \mathbb{R}^n$*.*

Proof. The first statement i) is a consequence of the formula (2.4) and the properties of the fundamental matrix. The second statement ii) follows from i) and the definition of $\mathcal{V}(t_0, t)$. From ii) (taking $y = 0$) and from the Definition 2.2, one obtains iii) and iv). □

Proposition 2.1. ([3], [8]) *Let us suppose that the matrices* $(M_\alpha(\cdot))_{\alpha=\overline{1,m}}$ *satisfy relations* (2.2), $\forall t \in D$, $\forall \alpha, \beta = \overline{1,m}$ *and let* $t_0 \in D$ *be a fixed point. For each* $v \in \mathbb{R}^n$ *and* $\alpha = \overline{1,m}$, *we define the functions*

$$u_{\alpha,v}\colon D \to \mathbb{R}^k, \quad u_{\alpha,v}(s) = N_\alpha^\top(s)\chi(t_0,s)^\top v, \quad \forall s \in D.$$

The following statements are equivalent:

i) *For each* $v \in \mathbb{R}^n$, $(u_{\alpha,v})_{\alpha=\overline{1,m}}$ *is a control for the PDE system* (2.1).

ii) *For each* $\alpha, \beta = \overline{1,m}$, *the following relations are satisfied on the* D*:*

$$\begin{aligned} &M_\alpha N_\beta N_\beta^\top + \frac{\partial N_\alpha}{\partial s^\beta} N_\alpha^\top + N_\alpha \frac{\partial N_\alpha^\top}{\partial s^\beta} + N_\beta N_\beta^\top M_\alpha^\top \\ &= M_\beta N_\alpha N_\alpha^\top + \frac{\partial N_\beta}{\partial s^\alpha} N_\beta^\top + N_\beta \frac{\partial N_\beta^\top}{\partial s^\alpha} + N_\alpha N_\alpha^\top M_\beta^\top. \end{aligned} \tag{2.5}$$

iii) *The curvilinear integral* $\int\limits_\gamma \chi(t_0,s) N_\alpha(s) N_\alpha^\top(s) \chi(t_0,s)^\top ds^\alpha$ *is path independent on* D.

2.2. **Quadratic affine forms on Hilbert spaces.** This section reconsiders two classical theorems regarding the quadratic affine forms on a Hilbert space, which we shall use further in order to obtain new results concerning the controllability of multitime PDE systems.

Theorem 2.2. *Let* $\mathcal{H}$ *be a real Hilbert space and* $T\colon \mathcal{H} \to \mathcal{H}$ *be a linear, continuous, autoadjoint, positive semidefinite operator. For each* $w \in \mathcal{H}$, *we consider the quadratic affine form*

$$F_w\colon \mathcal{H} \to \mathbb{R}, \quad F_w(v) = \langle T(v), v\rangle - 2\langle w, v\rangle, \quad \forall v \in \mathcal{H}.$$

i) *If* $v_0 \in \mathcal{H}$ *is a local minimum point of* F_w, *then* $T(v_0) = w$.

ii) *If* $v_0 \in \mathcal{H}$ *satisfies* $T(v_0) = w$, *then* v_0 *is a global minimum point of* F_w.

iii) *If there exists a local maximum point* $v_0 \in \mathcal{H}$ *for* F_w, *then* $w = 0$ *and* $T(v) = 0$, $\forall v \in \mathcal{H}$, *i.e., the function* F_w *is identically zero (and, obviously, each point is a global minimum point).*

iv) v_0 *is a local extremum point of* F_w *if and only if* v_0 *is a global minimum point for* F_w.

Theorem 2.3. *Let* $\mathcal{H}$ *be a real Hilbert space of finite dimension* n. *Let* $T\colon \mathcal{H} \to \mathcal{H}$ *be an autoadjoint, positive semidefinite, linear operator and let* A *be the matrix of* T, *relative to an arbitrary basis.*

For each $w \in \mathcal{H}$, *we consider the function*

$$F_w : \mathcal{H} \to \mathbb{R}, \quad F_w(v) = \langle T(v), v\rangle - 2\langle w, v\rangle, \quad \forall v \in \mathcal{H}.$$

Then, the following statements are equivalent:

i) T *is positive definite.*

ii) T *is bijective (equivalent to* $\operatorname{rank} A = n$*).*

iii) *For any* $w \in \mathcal{H}$, *the function* F_w *has at most a minimum point.*

iv) *For each* $w \in \mathcal{H}$, *the function* F_w *has at least a minimum point.*

v) *For each* $w \in \mathcal{H}$, *there exists a unique minimum point for* F_w.

Remark 2.1. According to Theorem 2.2, the minimum point in Theorem 2.3 may be understood either as local minimum point, global minimum point, or local extremum point.

3. INCREASING CURVES AND CURVILINEAR INTEGRALS

In this section, we define the notion of increasing curve (respectively, decreasing curve) and we prove some results related to it, needed for the further developement of our theory.

Lemma 3.1. *Let $f\colon [a,b] \to \mathbb{R}$ be a differentiable function such that, for each pair ξ_1, $\xi_2 \in [a,b]$, with $\xi_1 < \xi_2$, we have $f(\xi_1) < f(\xi_2)$ (i.e., f is strictly increasing). Suppose there exists $c \in [a,b]$, such that $f'(c) = 0$. Then there exists a sequence $\eta_p \in [a,b]$, with $f'(\eta_p) \neq 0$, $\forall p$, such that $\lim\limits_{p\to\infty} \eta_p = c$.*

Definition 3.1. Let $t_0, t \in \mathbb{R}^m$, such that $t_0^\alpha \leq t^\alpha$ (respectively $t_0^\alpha \geq t^\alpha$), $\forall \alpha = \overline{1,m}$ and let $\gamma\colon [a,b] \to \mathbb{R}^m$, with $\gamma(a) = t_0$, $\gamma(b) = t$ be a piecewise $\mathcal{C}^1$ curve. The curve γ between t_0 and t is called increasing (respectively decreasing) if, for each $\tau_1, \tau_2 \in [a,b]$, with $\tau_1 < \tau_2$, we have

$$\begin{cases} \gamma^\alpha(\tau_1) < \gamma^\alpha(\tau_2)\,(\text{respectively } \gamma^\alpha(\tau_1) > \gamma^\alpha(\tau_2)), & \text{if} \quad t_0^\alpha \neq t^\alpha; \\ \gamma^\alpha(\tau_1) = \gamma^\alpha(\tau_2), & \text{if} \quad t_0^\alpha = t^\alpha, \end{cases}$$

where $\gamma(\tau) = \Big(\gamma^\alpha(\tau)\Big)_{\alpha=\overline{1,m}}$.

Remark 3.1. Let $t_0, t \in D$, such that $t_0^\alpha \leq t^\alpha$ (respectively $t_0^\alpha \geq t^\alpha$), $\forall \alpha = \overline{1,m}$. There exists at least one increasing (respectively decreasing) $\mathcal{C}^1$ curve in D between t_0 and t. For example, the straight line segment $[t_0, t]$, parametrized by: $\gamma\colon [0,1] \to D$, $\gamma(\tau) = (1-\tau)t_0 + \tau t$, $\forall \tau \in [0,1]$, is increasing (respectively decreasing) and included in D (since D is a convex set).

Lemma 3.2. *Let $t_0, t \in D$, such that $t_0^\alpha \leq t^\alpha$ (respectively $t_0^\alpha \geq t^\alpha$), $\forall \alpha = \overline{1,m}$ and let $\gamma\colon [a,b] \to D$, with $\gamma(a) = t_0$, $\gamma(b) = t$ be a piecewise $\mathcal{C}^1$ increasing (respectively decreasing) curve from t_0 to t.*

If $P_1, P_2, \ldots, P_m : D \to [0,\infty)$ are continuous functions and $\int\limits_\gamma P_\alpha(s)\, ds^\alpha = 0$, then, for each $\alpha = \overline{1,m}$, with $t^\alpha \neq t_0^\alpha$, and for each $\tau \in [a,b]$, $P_\alpha(\gamma(\tau)) = 0$.

Proof. There exist the real numbers $\tau_0, \tau_1, \ldots, \tau_q$, such that

$$a = \tau_0 < \tau_1 < \cdots < \tau_q = b, \quad q \geq 1,\ q \in \mathbb{N},$$

and the curve γ is of class $\mathcal{C}^1$ on each subinterval $[\tau_j, \tau_{j+1}]$, $j = \overline{0, q-1}$.

We know that $\sum\limits_{j=0}^{q-1} \int\limits_{\tau_j}^{\tau_{j+1}} \sum\limits_{\alpha=1}^{m} P_\alpha(\gamma(\tau))\dot{\gamma^\alpha}(\tau)\, d\tau = 0$. But, for α satisfying $t^\alpha = t_0^\alpha$, the function $\gamma^\alpha(\cdot)$ is constant on $[\tau_0, \tau_q]$ (see the Definition 3.1). Consequently, $\dot{\gamma^\alpha}(\tau) = 0$, $\forall \tau \in [\tau_0, \tau_q]$, hence

$$\sum_{j=0}^{q-1} \int_{\tau_j}^{\tau_{j+1}} \sum_{\alpha \text{ with } t^\alpha \neq t_0^\alpha} P_\alpha(\gamma(\tau))\dot{\gamma^\alpha}(\tau)\, d\tau = 0.$$

Since γ^α is an increasing curve, it follows that $\dot{\gamma^\alpha}(\tau) \geq 0$, on each subinterval $[\tau_j, \tau_{j+1}]$. In conclusion each term of the foregoing sum is positive. In fact, each integral vanishes as term

in a sum equal to zero, i.e.,

$$\int_{\tau_j}^{\tau_{j+1}} \sum_{\alpha \text{ with } t^\alpha \neq t_0^\alpha} P_\alpha(\gamma(\tau))\dot{\gamma^\alpha}(\tau)\,\mathrm{d}\tau = 0, \quad \forall j = \overline{0, q-1}.$$

Moreover, since the integrands involved in the foregoing sum are positive functions, then, for each index $j = \overline{0, q-1}$, and for each α, satisfying $t^\alpha \neq t_0^\alpha$, we have $P_\alpha(\gamma(\tau))\dot{\gamma^\alpha}(\tau) = 0$, $\forall \tau \in [\tau_j, \tau_{j+1}]$. Hence, for each α, with $t^\alpha \neq t_0^\alpha$, we have

$$P_\alpha(\gamma(\tau)) = 0, \quad \forall \tau \in [\tau_j, \tau_{j+1}], \dot{\gamma^\alpha}(\tau) \neq 0. \qquad (**)$$

Let α be an arbitrary fixed index, satisfying $t^\alpha \neq t_0^\alpha$ and let $c \in [\tau_j, \tau_{j+1}]$, $\dot{\gamma^\alpha}(c) = 0$. By applying Lemma 3.1 for the function $\gamma^\alpha(\cdot)$, on the interval $[\tau_j, \tau_{j+1}]$ we obtain that there exists a sequence $\eta_p \in [\tau_j, \tau_{j+1}]$, with $\dot{\gamma^\alpha}(\eta_p) \neq 0, \forall p$, such that $\lim\limits_{p\to\infty} \eta_p = c$.

Since $\dot{\gamma^\alpha}(\eta_p) \neq 0$, according to $(**)$, we have $P_\alpha(\gamma(\eta_p)) = 0, \ \forall p$. Consequently

$$0 = \lim_{p\to\infty} P_\alpha(\gamma(\eta_p)) = P_\alpha(\gamma(c)).$$

We have proved by this that $P_\alpha(\gamma(c)) = 0, \ \forall c \in [\tau_j, \tau_{j+1}]$, with $\dot{\gamma^\alpha}(c) = 0$. Hence, via $(**)$, it follows $P_\alpha(\gamma(\tau)) = 0, \ \forall \tau \in [\tau_j, \tau_{j+1}]$ and, since $j = \overline{0, q-1}$ was arbitrary, we conclude that $P_\alpha(\gamma(\tau)) = 0, \forall \tau \in [a, b]$. □

4. IM-GRAMIAN SPACE AND CONDITIONS FOR MULTITIME CONTROLLABILITY

In [3], [8], the controllability gramian $\mathcal{C}(t_0, t)$ was defined whenever the relations (2.5) held. In this Section we shall extend the definition to a general situation, needing no more additional hypotheses. But, unlike the previous definition, our extended gramian depends on the curve γ too, resulting the notion of γ - gramian matrix. Accordingly, instead of the controllability gramian image we shall introduce the Im - gramian space $\mathcal{W}(t_0, t)$. The result is a necessary condition for controllability expressed using the space $\mathcal{W}(t_0, t)$.

Definition 4.1. Let us suppose that the matrices $M_\alpha(\cdot)$ verify the relations (2.2).

i) Let $\gamma\colon [a, b] \to D$ be a piecewise $\mathcal{C}^1$ curve, with fixed origin $t_0 = \gamma(a)$. The matrix

$$\mathcal{C}_\gamma := \int_\gamma \chi(t_0, s) N_\alpha(s) N_\alpha^\top(s) \chi(t_0, s)^\top \mathrm{d}s^\alpha$$

is called *γ-gramian matrix* associated to the PDE system (2.1).

ii) Suppose that, for each $\alpha, \beta = \overline{1, m}$, relations (2.5) are true. According to Proposition 2.1, in this case, the curvilinear integral from i) depends only on the ends points and not on the curve joining them. Let $t_0, t \in D$ and $\gamma_{t_0,t} : [a, b] \to D$ be a piecewise $\mathcal{C}^1$ curve, with $\gamma_{t_0,t}(a) = t_0$ and $\gamma_{t_0,t}(b) = t$. The matrix

$$\mathcal{C}(t_0, t) := \int_{\gamma_{t_0,t}} \chi(t_0, s) N_\alpha(s) N_\alpha^\top(s) \chi(t_0, s)^\top \mathrm{d}s^\alpha$$

is called the controllability gramian.

Remark 4.1. Let us suppose that the matrices $M_\alpha(\cdot)$ satisfy the relations (2.2).

Let $\gamma\colon [a,b] \to D$ be a piecewise $\mathcal{C}^1$ curve, and $t_0 := \gamma(a)$, $t := \gamma(b)$.

Let $\gamma^-\colon [a,b] \to D$, $\gamma^-(\tau) = \gamma(a+b-\tau)$, $\forall \tau \in [a,b]$. Obviously, we have $\gamma^-(a) = t$, $\gamma^-(b) = t_0$.

One verifies immediately

$$\chi(t,t_0)\mathcal{C}_\gamma \chi(t,t_0)^\top = -\mathcal{C}_{\gamma^-}, \quad \text{hence} \quad \operatorname{rank}(\mathcal{C}_\gamma) = \operatorname{rank}(\mathcal{C}_{\gamma^-}).$$

Knowing that $\chi(t,t_0)^\top$ is invertible, it follows the equality $\operatorname{Im}(\mathcal{C}_{\gamma^-}) = \chi(t,t_0)\operatorname{Im}(\mathcal{C}_\gamma)$.

Remark 4.2. Let $t_0, t \in \mathbb{R}^m$, such that $t_0^\alpha \le t^\alpha$ (respectively $t_0^\alpha \ge t^\alpha$), $\forall \alpha = \overline{1,m}$ and let $\gamma\colon [a,b] \to \mathbb{R}^m$, $\gamma(a) = t_0$, $\gamma(b) = t$ be a piecewise $\mathcal{C}^1$ curve.

One remarks that γ is an increasing (respectively decreasing) curve between t_0 to t, if and only if γ^- is a decreasing (respectively increasing) curve between t_0 to t.

Definition 4.2. Suppose that the matrices $M_\alpha(\cdot)$ satsfy relations (2.2). Let $t_0, t \in D$, such that $t_0^\alpha \le t^\alpha$ (respectively $t_0^\alpha \ge t^\alpha$), $\forall \alpha = \overline{1,m}$. We consider the set

$$\mathcal{W}(t_0,t) := \bigcap_{\gamma_{t_0,t}} \operatorname{Im}(\mathcal{C}_{\gamma_{t_0,t}}) \tag{4.1}$$

where the intersection is taken over all piecewise $\mathcal{C}^1$ increasing (respectively decreasing) curves $\gamma_{t_0,t}$ in D. The set $\mathcal{W}(t_0,t)$ is a vector subspace of $\mathbb{R}^n$ and we shall call it the Im - gramian space.

Remark 4.3. Using Remark 4.1 and Remark 4.2, we write the conditions in Definition 4.2 as

$$\mathcal{W}(t,t_0) = \chi(t,t_0)\mathcal{W}(t_0,t). \tag{4.2}$$

Remark 4.4. If for each $\alpha, \beta = \overline{1,m}$, the relations (2.5) are true, then $\mathcal{W}(t_0,t) = \operatorname{Im}(\mathcal{C}(t_0,t))$ (see also the Definition 4.1).

Proposition 4.1. *Suppose the matrices $M_\alpha(\cdot)$ satisfy relations (2.2). Let $t_0, t \in D$, such that $t_0^\alpha \le t^\alpha$ (respectively, $t_0^\alpha \ge t^\alpha$), $\forall \alpha = \overline{1,m}$. Then*

$$\mathcal{V}(t_0,t) \subseteq \mathcal{W}(t_0,t). \tag{4.3}$$

Proof. Let γ be a piecewise $\mathcal{C}^1$ increasing (respectively decreasing) curve between t_0 and t in D. It is enough to prove the inclusion

$$\mathcal{V}(t_0,t) \subseteq \operatorname{Im}(\mathcal{C}_\gamma), \tag{4.4}$$

or, equivalent, $(\mathcal{V}(t_0,t))^\perp \supseteq (\operatorname{Im}(\mathcal{C}_\gamma))^\perp = \operatorname{Ker}((\mathcal{C}_\gamma)^\top)$.

We have $v \in \operatorname{Ker}((\mathcal{C}_\gamma)^\top) \iff ((\mathcal{C}_\gamma)^\top)v = 0 \iff v^\top \mathcal{C}_\gamma = 0$. Hence $v^\top \mathcal{C}_\gamma v = 0$, or

$$\int_\gamma v^\top \chi(t_0,s) N_\alpha(s) N_\alpha^\top(s) \chi(t_0,s)^\top v \, \mathrm{d}s^\alpha = \int_\gamma \left\| v^\top \chi(t_0,s) N_\alpha(s) \right\|^2 \mathrm{d}s^\alpha = 0.$$

Applying Lemma 3.2 for $P_\alpha(s) = \left\| v^\top \chi(t_0,s) N_\alpha(s) \right\|^2$ we conclude that, for each α, with $t^\alpha \neq t_0^\alpha$ and for each τ, $v^\top \chi(t_0,\gamma(\tau)) N_\alpha(\gamma(\tau)) = 0$.

Let $(u_\alpha(\cdot))_{\alpha=\overline{1,m}}$ be an arbitrary control. It follows

$$\left\langle \int_\gamma \chi(t_0,s) N_\alpha(s) u_\alpha(s) \, \mathrm{d}s^\alpha \, ; \, v \right\rangle = v^\top \int_\gamma \chi(t_0,s) N_\alpha(s) u_\alpha(s) \, \mathrm{d}s^\alpha$$

$$= \int_{\gamma} \sum_{\alpha \text{ with } t^\alpha \neq t_0^\alpha} v^\top \chi(t_0, s) N_\alpha(s) u_\alpha(s)\, \mathrm{d}s^\alpha = 0.$$

Hence $v \in (\mathcal{V}(t_0, t))^\perp$, proving the inclusion $\mathrm{Ker}((\mathcal{C}_\gamma)^\top) \subseteq (\mathcal{V}(t_0, t))^\perp$. □

The following Theorem, giving necessary conditions for controllability, is an immediate consequence of Theorem 2.1 and Proposition 4.1.

Theorem 4.1. *Let us consider the PDE system* (2.1)*, with the matrix functions* $M_\alpha(\cdot)$ *satisfying the relations* (2.2)*.*

i) *Let* $t_0, t \in D$*, such that* $t_0^\alpha \leq t^\alpha$ *(respectively,* $t_0^\alpha \geq t^\alpha$*),* $\forall \alpha = \overline{1, m}$*. If the phase* (t_0, x_0) *transfers into the phase* (t, y)*, then*

$$x_0 - \chi(t_0, t) y \in \mathcal{W}(t_0, t).$$

ii) *If the phase* (t_0, x_0) *is controllable, then* $\exists t \in D$*, with* $t^\alpha > t_0^\alpha$*,* $\forall \alpha$*, such that* $x_0 \in \mathcal{W}(t_0, t)$*.*

iii) *Let* $t_0, t \in D$*, with* $t_0^\alpha < t^\alpha$*,* $\forall \alpha$*. If the PDE system* (2.1) *is completely controllable from the multitime* t_0 *into the multitime* t*, then* $\mathcal{W}(t_0, t) = \mathbb{R}^n$*.*

Remark 4.5. When adding relations (2.5) to the hypotheses of Proposition 4.1, the inclusion (4.4) becomes equality ([3]), i.e., $\mathcal{V}(t_0, t) = \mathrm{Im}(\mathcal{C}(t_0, t)) = \mathcal{W}(t_0, t)$. These allow us to derive necessary and sufficient conditions for controllability.

Generally, the inclusion (4.3) is strict and the following example comes to justify this statement:

Example 4.1. We consider: $m = 2$, $n = 2$, $k = 2$,

$$M_1 = M_2 = 0, \qquad N_2 = 0, \qquad N_1(s^1, s^2) = \begin{pmatrix} s^2 & 0 \\ 0 & s^2 \end{pmatrix} = s^2 I_2,$$

$$\text{a) } D = \mathbb{R}^2; \qquad \text{b) } D = \left\{ \left(t^1, t^2\right) \in \mathbb{R}^2 \,\middle|\, t^2 > 0 \right\} = \mathbb{R} \times (0, \infty).$$

We shall prove that, for each $t_0, t \in D$, with $t_0^1 < t^1$, $t_0^2 < t^2$ we have

a) $\mathcal{V}(t_0, t) = 0$ and $\mathcal{W}(t_0, t) = \mathbb{R}^2 = \mathcal{M}_{2,1}(\mathbb{R})$ and

b) $\mathcal{V}(t_0, t) = \mathcal{W}(t_0, t) = \mathbb{R}^2$.

Moreover, $\chi(s_0, s) = I_2$, $\forall (s_0, s) \in \mathbb{R}^2 \times \mathbb{R}^2$.

a) Indeed, according to Definition 2.1, the pair (u_1, u_2) is a control if and only if relation (2.3) holds, i.e.,

$$\frac{\partial}{\partial s^2}\left(N_1(s^1, s^2)\right) u_1(s^1, s^2) + N_1(s^1, s^2) \frac{\partial}{\partial s^2}\left(u_1(s^1, s^2)\right) = 0, \quad \forall (s^1, s^2) \in \mathbb{R}^2,$$

or

$$\frac{\partial}{\partial s^2}\left(s^2 u_1(s^1, s^2)\right) = 0, \quad \forall (s^1, s^2) \in \mathbb{R}^2, \tag{4.5}$$

equivalent to the existence of a $\mathcal{C}^1$ function $f : \mathbb{R} \to \mathbb{R}^2 = \mathcal{M}_{2,1}(\mathbb{R})$, such that

$$s^2 u_1(s^1, s^2) = f(s^1), \quad \forall (s^1, s^2) \in \mathbb{R}^2.$$

Setting $s^2 = 0$, we obtain $f(s^1) = 0$, $\forall s^1 \in \mathbb{R}$. Hence

$$s^2 u_1(s^1, s^2) = 0, \quad \forall (s^1, s^2) \in \mathbb{R}^2,$$

and it follows $u_1(s^1, s^2) = 0$, $\forall (s^1, s^2) \in \mathbb{R}^2$, $s^2 \neq 0$. Moreover, the continuity of u_1, ensures us that

$$u_1(s^1, s^2) = 0, \quad \forall (s^1, s^2) \in \mathbb{R}^2.$$

For $u_1 = 0$ and for u_2 arbitrary, relation (4.5) is satisfied.

In conclusion, the set of controls contains pairs (u_1, u_2), with $u_1 = 0$ and $u_2 : \mathbb{R}^2 \to \mathbb{R}^2 = \mathcal{M}_{2,1}(\mathbb{R})$ an arbitrary $\mathcal{C}^1$ function.

Let γ be a piecewise $\mathcal{C}^1$ increasing curve between t_0 and t. Since u_1 and N_2 vanish, it follows

$$\int_\gamma \chi(t_0, s) N_1(s) u_1(s) \mathrm{d}s^1 + \chi(t_0, s) N_2(s) u_2(s) \mathrm{d}s^2 = 0,$$

hence $\mathcal{V}(t_0, t) = 0$ (in fact, since the curvilinear integral is path independent, the assumption related to the increasing character of the curve γ is unnecessary).

According to the definition, $\mathcal{C}_\gamma = \int_\gamma (s^2)^2 I_2 \mathrm{d}s^1 = \Big(\int_\gamma (s^2)^2 \mathrm{d}s^1 \Big) I_2$. We prove that $\int_\gamma (s^2)^2 \mathrm{d}s^1 \neq 0$. Indeed, suppose that $\int_\gamma (s^2)^2 \mathrm{d}s^1 = 0$. Applying Lemma 3.2, we obtain $(\gamma^2(\tau))^2 = 0$, $\forall \tau$, hence γ^2 is constant (zero) and $t_0^2 = t^2$, which is false.

It follows that the matrix $\mathcal{C}_\gamma$ is of rank 2 and $\mathrm{Im}(\mathcal{C}_\gamma) = \mathbb{R}^2$, therefore $\mathcal{W}(t_0, t) = \mathbb{R}^2$.

b) Similar to case a), we obtain that (u_1, u_2) is a control if and only if relation (4.5) holds on the set D. Let $v \in \mathbb{R}^2 = \mathcal{M}_{2,1}(\mathbb{R})$. We set

$$u_1(s^1, s^2) = \frac{1}{s^2 (t^1 - t_0^1)} \cdot v, \quad u_2(s^1, s^2) = 0, \quad \forall (s^1, s^2) \in \mathbb{R} \times (0, \infty) = D.$$

Then, relation (4.5) is satisfied for each pair (u_1, u_2), hence (u_1, u_2) is a control.

Let γ be a piecewise $\mathcal{C}^1$ curve, included in D, connecting t_0 and t. We shall determine next $\mathcal{V}(t_0, t)$. We have

$$\int_\gamma \chi(t_0, s) N_1(s) u_1(s) \mathrm{d}s^1 + \chi(t_0, s) N_2(s) u_2(s) \mathrm{d}s^2 = v.$$

Since v is arbitrary, it follows $\mathcal{V}(t_0, t) = \mathbb{R}^2$, and, using (4.3), we obtain $\mathcal{W}(t_0, t) = \mathbb{R}^2$.

Remark 4.6. In the previous Example, case a) $D = \mathbb{R}^2$, the inclusion (4.3) is strict. Also, we remark that the converse of the Theorem 4.1 is not always true. Indeed, for each $t_0, t \in \mathbb{R}^2$, with $t_0^1 < t^1$, $t_0^2 < t^2$, we have $\mathcal{V}(t_0, t) = 0$ and using Theorem 2.1, iii) we conclude that no state (t_0, x_0), with $x_0 \neq 0$, is controllable; However $\mathcal{W}(t_0, t) = \mathbb{R}^2 = \mathbb{R}^n$.

Switching to case b), $D = \mathbb{R} \times (0, \infty)$, for each $t_0, t \in \mathbb{R}^2$, with $t_0^1 < t^1$, $t_0^2 < t^2$, we have $\mathcal{V}(t_0, t) = \mathbb{R}^2 = \mathbb{R}^n$ and, according the Theorem 2.1, iv), the system is completely controllable.

Therefore, the domain of the matrix functions $M_\alpha(\cdot)$ and $N_\alpha(\cdot)$ is an important ingredient for controllability.

5. DECISION FUNCTIONAL FOR MULTITIME CONTROLLABILITY

As we know, two type of methods for analyzing controllability are of great importance, namely direct ones and dual ones.

Suppose that the matrices $M_\alpha(\cdot)$ satisfy the relations (2.2). We consider the adjoint (dual) PDE system associated to PDE system (2.1), i.e.,

$$\frac{\partial \varphi}{\partial t^\alpha} = -(M_\alpha(t))^\top \varphi, \quad \forall \alpha = \overline{1,m}. \tag{5.1}$$

Let $t_0 \in D$ and $v \in \mathbb{R}^n$. The Cauchy problem $\{(5.1),\, \varphi(t_0) = v\}$ has the solution

$$\varphi_v(\,\cdot\,, t_0) : D \to \mathbb{R}^n, \quad \varphi_v(s, t_0) = \chi(t_0, s)^\top v, \quad \forall s \in D. \tag{5.2}$$

Let us denote by $\mathcal{S}$, the set of solutions of the PDE system (5.1), i.e.,

$$\mathcal{S} = \left\{ \varphi \colon D \to \mathbb{R}^n \;\middle|\; \varphi \text{ solution in } (5.1) \right\} = \left\{ \varphi_v(\,\cdot\,, t_0) \;\middle|\; v \in \mathbb{R}^n \right\}.$$

First, we remark that $\mathcal{S}$ is a real vector space.

Let $x_0 \in \mathbb{R}^n$, $t_0, t \in D$, and $\gamma_{t_0,t}$ be a piecewise $\mathcal{C}^1$ curve between t_0 and t in D. We define the functional $F_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t) : \mathcal{S} \to \mathbb{R}$,

$$F_{\gamma_{t_0,t}}(\varphi, x_0; t_0, t) = \int\limits_{\gamma_{t_0,t}} \left\| N_\alpha(s)^\top \varphi(s) \right\|^2 \mathrm{d}s^\alpha - 2\langle x_0, \varphi(t_0) \rangle, \quad \forall \varphi \in \mathcal{S}, \tag{5.3}$$

called *the controllability $\gamma_{t_0,t}$-functional*.

Obviously, the function $L_{t_0} : \mathbb{R}^n \to \mathcal{S}$, $L_{t_0}(v) = \varphi_v(\,\cdot\,, t_0)$ is a vector space isomorphism. Its inverse is $L_{t_0}^{-1} : \mathcal{S} \to \mathbb{R}^n$, $L_{t_0}^{-1}(\varphi) = \varphi(t_0)$, $\forall \varphi \in \mathcal{S}$.

We define next the function

$$\widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t) \colon \mathbb{R}^n \to \mathbb{R}, \quad \widetilde{F}_{\gamma_{t_0,t}}(v, x_0; t_0, t) = F_{\gamma_{t_0,t}}(L_{t_0}(v), x_0; t_0, t). \tag{5.4}$$

We rewrite $\widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ in order to obtain a quadratic affine form as

$$\widetilde{F}_{\gamma_{t_0,t}}(v, x_0; t_0, t) = \left\langle \mathcal{C}_{\gamma_{t_0,t}} v\,, v \right\rangle - 2\left\langle x_0, v \right\rangle, \quad \forall v \in \mathbb{R}^n. \tag{5.5}$$

Remark 5.1. From (5.4) and from the fact that L_{t_0} is a bijective function, it derives that v is an extremum point (respectively minimum, maximum) for $\widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ if and only if $L_{t_0}(v) = \varphi_v(\,\cdot\,, t_0)$ is an extremum point (respectively minimum, maximum) for $F_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$. Conversely: φ is an extremum point (respectively minimum, maximum) for $F_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ if and only if $L_{t_0}^{-1}(\varphi) = \varphi(t_0)$ is an extremum point (respectively minimum, maximum) for $\widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$. Hence L_{t_0} is a bijection between the extremum point set (respectively minimum, maximum) of $\widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ and the extremum point set (respectively minimum, maximum) of $F_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$. The notions of extremum (respectively minimum, maximum) are understood globally, for the time being.

In order to work with local extremum points, we need to define a topology on $\mathcal{S}$. Since L_{t_0} is an isomorphism between vector spaces, it follows that $\mathcal{S}$ has finite dimension, namely n. We endow $\mathcal{S}$ with the topology induced by an arbitrary norm. Such norm always exists on $\mathcal{S}$; for example, $\|\varphi\|_{\mathcal{S}} = \|L_{t_0}^{-1}(\varphi)\| = \|\varphi(t_0)\|$. Since $\mathcal{S}$ has finite dimension, each two norms are equivalent, hence they induce the same topology. Now, since L_{t_0} is a homeomorphism between $\mathbb{R}^n$ and $\mathcal{S}$, it follows that L_{t_0} is a bijection between the set of local extremum points (respectively local minimum, local maxim) of $\widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ and the set of local extremum points (respectively local minimum, local maxim) of $F_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$.

In the sequel, we suppose again that the matrices $M_\alpha(\cdot)$ satisfy relations (2.2) on D.

Let $t_0, t \in D$, with $t_0^\alpha \leq t^\alpha$, $\forall \alpha$ and let $\gamma_{t_0,t}\colon [a,b] \to D$ be a piecewise $\mathcal{C}^1$ increasing curve, running from t_0 to t. Then

$$\left\langle \mathcal{C}_{\gamma_{t_0,t}} v\,, v \right\rangle = \int\limits_{\gamma_{t_0,t}} \left\| N_\alpha(s)^\top \chi(t_0,s)^\top v \right\|^2 \mathrm{d}s^\alpha \geq 0, \quad \forall v \in \mathbb{R}^n, \tag{5.6}$$

The inequality (5.6) holds since $\dot{\gamma}_{t_0,t}^\alpha(\tau) \geq 0$, $\forall \tau$ (eventually, for a finite number of points, we have lateral derivative).

The matrix $\mathcal{C}_{\gamma_{t_0,t}}$ is symmetric. Using relation (5.6), we conclude that the function $\widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ has the expression of F_w, as in Theorems 2.2 and 2.3. Here, $\mathcal{H} = \mathbb{R}^n$, $\dim \mathcal{H} = n < \infty$, $T(v) = \mathcal{C}_{\gamma_{t_0,t}} v$; T is linear, hence continuous (since $\mathcal{H}$ is of finite dimension); T is autoadjoint and positive semidefinite (see (5.6)). Also $w = x_0$.

Consequently, we may apply Theorems 2.2 and 2.3, for $\widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$, resulting that the local extremum points of $\widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$, if they exist, are in fact global minimum points. According the Remark 5.1, same thing happens for $F_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ too. Therefore, from now on, we shall refer only to (global) minimum points.

The following result is a consequence of Theorem 2.2 and Remark 5.1.

Proposition 5.1. *Let $t_0, t \in D$, with $t_0^\alpha \leq t^\alpha$, $\forall \alpha$, let $\gamma_{t_0,t}\colon [a,b] \to D$ be a piecewise $\mathcal{C}^1$ increasing curve between t_0 and t, and let $x_0 \in \mathbb{R}^n$. The following statements are equivalent:*

i) *$F_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ admits at least one minimum point.*

ii) *$\widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ admits at least a minimum point.*

iii) *$x_0 \in \mathrm{Im}(\mathcal{C}_{\gamma_{t_0,t}})$.*

Similarly, the next Proposition is the result of combining Theorem 2.3 and Remark 5.1.

Proposition 5.2. *Let $t_0, t \in D$, with $t_0^\alpha \leq t^\alpha$, $\forall \alpha$ and let $\gamma_{t_0,t}\colon [a,b] \to D$ be a piecewise $\mathcal{C}^1$ increasing curve. The following statements are equivalent:*

i) *For each $x_0 \in \mathbb{R}^n$, there exists a unique minimum point for $F_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$.*

ii) *For each $x_0 \in \mathbb{R}^n$, there exists a unique minimum point for $\widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$.*

iii) $\mathrm{rank}\,(\mathcal{C}_{\gamma_{t_0,t}}) = n$.

Combining Propositions 5.1, 5.2 and Theorem 4.1 we obtain the following Corollary.

Theorem 5.1. *Let us consider the PDE system* (2.1)*, with the matrix functions $M_\alpha(\cdot)$ verifying the relations* (2.2)*.*

i) *Let $t_0, t \in D$, such that $t_0^\alpha \leq t^\alpha$, $\forall \alpha = \overline{1,m}$. If the phase (t_0, x_0) transfers into the phase (t, y), then for each piecewise $\mathcal{C}^1$ increasing curve $\gamma_{t_0,t}$, both $F_{\gamma_{t_0,t}}(\,\cdot\,, x_0 - \chi(t_0,t)y; t_0, t)$ and $\widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0 - \chi(t_0,t)y; t_0, t)$ have global minimum points.*

ii) *If the phase (t_0, x_0) is controllable, then $\exists t \in D$, with $t^\alpha > t_0^\alpha$, $\forall \alpha$, such that for each piecewise $\mathcal{C}^1$ increasing curve $\gamma_{t_0,t}$, $F_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ and $\widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ have global minimum points.*

iii) *Let $t_0, t \in D$, with $t_0^\alpha < t^\alpha$, $\forall \alpha$. If the PDE system* (2.1) *is completely controllable from t_0 to t, then for each piecewise $\mathcal{C}^1$ increasing curve $\gamma_{t_0,t}$ and each $x_0 \in \mathbb{R}^n$, there exists a unique (global) minimum point for $F_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$.*

Similar statements are valid for $\widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ too.

6. CONTROLLABILITY FUNCTIONAL

We consider the PDE system (2.1), with $M_\alpha(\cdot)$ and $N_\alpha(\cdot)$ satisfying relations (2.2) and (2.5) on D.

Let $x_0 \in \mathbb{R}^n$ and $t_0, t \in D$. Let $\gamma_{t_0,t}$ be a piecewise $\mathcal{C}^1$ curve on D. According to (5.5), we have

$$\widetilde{F}_{\gamma_{t_0,t}}(v, x_0; t_0, t) = \left\langle \mathcal{C}_{\gamma_{t_0,t}} v\,, v \right\rangle - 2\left\langle x_0, v \right\rangle, \quad \forall v \in \mathbb{R}^n.$$

Proposition 2.1, Definition 4.1 proved that, in this case, the functional $\mathcal{C}_{\gamma_{t_0,t}}$ does not depend on the curve $\gamma_{t_0,t}$, but only on the endpoints t_0, t. Therefore, we may denote it by $\mathcal{C}(t_0, t)$, and using (5.5), it follows that $\widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ also depends only on the endpoints t_0, t. We denote

$$\widetilde{F}(v, x_0; t_0, t) := \widetilde{F}_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t) = \langle \mathcal{C}(t_0, t)v\,, v\rangle - 2\left\langle x_0, v\right\rangle, \quad \forall v \in \mathbb{R}^n. \tag{6.1}$$

Since

$$F_{\gamma_{t_0,t}}(\varphi, x_0; t_0, t) = \widetilde{F}_{\gamma_{t_0,t}}(L_{t_0}^{-1}(\varphi), x_0; t_0, t) = \widetilde{F}(L_{t_0}^{-1}(\varphi), x_0; t_0, t), \quad \forall \varphi \in \mathcal{S},$$

it follows that $F_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ depends only on the endpoints t_0, t, and not on the curve $\gamma_{t_0,t}$. We denote

$$F(\,\cdot\,, x_0; t_0, t) := F_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t),$$

and we call it the *controllability functional*.

Let $t_0, t \in D$, with $t_0^\alpha < t^\alpha$, $\forall \alpha$. From Theorem 2.1 and Remark 4.5, it follows that if relations (2.5) hold, then the PDE system (2.1) is completely controllable from t_0 to t if and only if $\operatorname{rank} \mathcal{C}(t_0, t) = n$; also, as consequence of Theorem 2.2, Theorem 2.3, and Remark 5.1, we may state the following result.

Theorem 6.1. *Let us consider the PDE system* (2.1)*, with $M_\alpha(\cdot)$, $N_\alpha(\cdot)$ satisfying relations* (2.2) *and* (2.5) *on D.*

i) *Let $t_0, t \in D$, $t_0 \neq t$, $t_0^\alpha \leq t^\alpha$, $\forall \alpha$. Then the phase (t_0, x_0) transfers to the phase (t, y) if and only if $F(\,\cdot\,, x_0 - \chi(t_0, t)y; t_0, t)$ has at least a minimum point.*

ii) *The phase (t_0, x_0) is controllable if and only if $\exists t \in D$, with $t^\alpha > t_0^\alpha$, $\forall \alpha$, such that $F(\,\cdot\,, x_0; t_0, t)$ has at least a minimum point.*

iii) *Let $t_0, t \in D$, with $t_0^\alpha < t^\alpha$, $\forall \alpha$. The PDE system* (2.1) *is completely controllable from t_0 to t if and only if, for each $x_0 \in \mathbb{R}^n$, there exists a unique global minimum point for the functional $F(\,\cdot\,, x_0; t_0, t)$.*

iv) *Let $t_0, t \in D$, with $t_0^\alpha < t^\alpha$, $\forall \alpha$. The PDE system* (2.1) *is completely controllable from t_0 to t if and only if, for each $x_0 \in \mathbb{R}^n$, the functional $F(\,\cdot\,, x_0; t_0, t)$ has at least a minimum point.*

v) *Let $t_0, t \in D$, with $t_0^\alpha < t^\alpha$, $\forall \alpha$. The PDE system* (2.1) *is completely controllable from t_0 to t if and only if, for each $x_0 \in \mathbb{R}^n$, the functional $F(\,\cdot\,, x_0; t_0, t)$ has at most a minimum point.*

Analogously for $\widetilde{F}(\,\cdot\,, x_0; t_0, t)$.

7. UNBOUNDED EXTENSION OF CONTROLLABILITY FUNCTIONAL

Let us give an example of a complete controllable PDE system, with the matrices M_α, N_α, verifying the relations (2.2) and (2.5). Theorem 5.1 proved that, for t_0, t, with $t_0^\alpha < t^\alpha$ and for each increasing curve $\gamma_{t_0,t}$ from t_0 to t, the controllability $\gamma_{t_0,t}$ - functional has a unique global minimum point on the set $\mathcal{S}$.

Moreover, formula (5.3) defining the functional $F_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ may be also extended to a space $\mathcal{S}_1$ which includes $\mathcal{S}$, as, for example, $\mathcal{C}^r(D;\mathbb{R}^n)$ with $r \in \{0,1,2\}$. This means that we can define a new functional

$$J_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t) : \mathcal{S}_1 \to \mathbb{R},$$

$$J_{\gamma_{t_0,t}}(\varphi, x_0; t_0, t) = \int\limits_{\gamma_{t_0,t}} \left\| N_\alpha(s)^\top \varphi(s) \right\|^2 \mathrm{d}s^\alpha - 2\langle x_0, \varphi(t_0)\rangle, \quad \forall \varphi \in \mathcal{S}_1. \tag{7.1}$$

The functional $J_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ is obviously an extension of $F_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$.

We select $\mathcal{S}_1 = \mathcal{C}^2(D;\mathbb{R}^n)$. For the PDE system in our example, we shall prove that, for each $x_0 \neq 0$, for each t_0, t with $t_0^\alpha < t^\alpha$, $\forall \alpha$ and for each increasing curve $\gamma_{t_0,t}$ between t_0 and t, the extension $J_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ is unbounded from below. Hence, Theorems 5.1 and 6.1 are no longer valid on spaces strictly including the set $\mathcal{S}$.

Suppose $m = 2,\ n = 2,\ k = 1,\ D = \mathbb{R}^2$,

$$M_1(s) = M_2(s) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ N_1(s) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ N_2(s) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \forall s \in \mathbb{R}^2.$$

The relations (2.2) and (2.5) hold. We have $\chi(s_1, s_2) = I_2$, $\forall s_1, s_2 \in \mathbb{R}^2$.

Let $t_0 = (t_0^1, t_0^2)$, $t = (t^1, t^2)$, with $t_0^1 < t^1$, $t_0^2 < t^2$ and $x_0 = (a, b)^\top$. For $v = (v_1, v_2)^\top \in \mathbb{R}^2 = \mathcal{M}_{2,1}(\mathbb{R})$, we have $\varphi_v(s, t_0) = \chi(t_0, s)^\top v = v$, $\forall s \in \mathbb{R}^2$; $N_1^\top(s)\varphi_v(s, t_0) = v_1$, $\quad N_2^\top(s)\varphi_v(s, t_0) = v_2$, $\quad \forall s \in \mathbb{R}^2$; $F(\,\cdot\,, x_0; t_0, t) : \mathcal{S} \to \mathbb{R}$, $F\Big(\varphi_v(\,\cdot\,, t_0), x_0; t_0, t\Big) = (t^1 - t_0^1) \cdot \left(v_1 - \dfrac{a}{t^1 - t_0^1}\right)^2 + (t^2 - t_0^2) \cdot \left(v_2 - \dfrac{b}{t^2 - t_0^2}\right)^2 - \dfrac{a^2}{t^1 - t_0^1} - \dfrac{b^2}{t^2 - t_0^2} \geq - \dfrac{a^2}{t^1 - t_0^1} - \dfrac{b^2}{t^2 - t_0^2}$. The equality holds if and only if $v_1 = \dfrac{a}{t^1 - t_0^1}$ and $v_2 = \dfrac{b}{t^2 - t_0^2}$.

Hence, for each $t_0 = (t_0^1, t_0^2)$, $t = (t^1, t^2)$ with $t_0^1 < t^1$, $t_0^2 < t^2$ and each $x_0 = (a, b)^\top$, the functional $F(\,\cdot\,, x_0; t_0, t)$ has a unique global minimum point, that is $\varphi_{v_0}(\,\cdot\,, t_0)$, where $v_0 = \left(\dfrac{a}{t^1 - t_0^1}, \dfrac{b}{t^2 - t_0^2}\right)$. Then, Theorem 6.1 ensures us that the system is complete controllable.

We denote $\mathcal{S}_1 = \left\{\varphi\colon \mathbb{R}^2 = D \to \mathbb{R}^2 \;\middle|\; \varphi \text{ of class } \mathcal{C}^2\right\}$. Obviously $\mathcal{S} \subseteq \mathcal{S}_1$.

Let $\gamma_{t_0,t}$ be an increasing curve from t_0 to t. We choose $x_0 = (a, b)^\top \neq (0, 0)^\top$. Let us define $J_{\gamma_{t_0,t}}(\,\cdot\,, x_0; t_0, t)$ by the formula (7.1). We obtain an extension of the controllability functional to the space $\mathcal{S}_1$.

Let $q > 0$. If $a + b \neq 0$, we select $c = a + b$. If $b = -a$, we select $c = a$; since $x_0 \neq 0$, we have $c \neq 0$. We choose $\varphi_1(s^1, s^2) = c\sqrt{\dfrac{q}{1 + q^2(s^1 + s^2 - t_0^1 - t_0^2)^2}}$, $\forall (s^1, s^2) \in \mathbb{R}^2$. If $a + b \neq 0$, we consider the function $\varphi_2(\cdot) = \varphi_1(\cdot)$. If $b = -a$, we consider the function $\varphi_2(\cdot) = -\varphi_1(\cdot)$. We select $\varphi(\cdot) = (\varphi_1(\cdot), \varphi_2(\cdot))^\top$. Obviously, $\varphi \in \mathcal{S}_1$. If $a + b \neq 0$ we obtain $J_{\gamma_{t_0,t}}(\varphi, x_0; t_0, t) = (a + b)^2 \arctan\Big(q(t^1 + t^2 - t_0^1 - t_0^2)\Big) - 2(a + b)^2\sqrt{q}$. If $b = -a \neq 0$ we obtain $J_{\gamma_{t_0,t}}(\varphi, x_0; t_0, t) = a^2 \arctan\Big(q(t^1 + t^2 - t_0^1 - t_0^2)\Big) - 4a^2\sqrt{q}$. Both times we have $\lim\limits_{q\to\infty} J_{\gamma_{t_0,t}}(\varphi, x_0; t_0, t) = -\infty$.

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