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FIXED AND STATIONARY POINTS OF GENERALIZED WEAK CONTRACTIONS

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Abstract. In this paper some results on fixed and stationary points on very general settings have been obtained. The results obtained herein extend and unify some recent results.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper $\mathbb{N}$ denotes the set of natural numbers, X a metric space (X, d), $CL(X)$ the collection of all nonempty closed subsets of X, $CB(X)$ the collection of all nonempty closed and bounded subsets of X and H the Hausdorff metric induced by d, i.e.,

$$H(A,B) = \begin{cases} \max\left\{\sup\limits_{x\in A} d(x,B),\ \sup\limits_{y\in B} d(y,A)\right\}, & \text{if the maximum exists} \\ \infty, & \text{otherwise} \end{cases}$$

for all $A, B \subseteq CL(X)$, where $d(x,B) = \inf\limits_{y\in B} d(x,y)$.

Let $S\colon X \to X$ and $T\colon X \to CL(X)$. A point $z \in X$ is a coincidence point of the mappings S and T if $Sz \in Tz$ and a fixed point if $z = Sz \in Tz$. Further, a point $z \in X$ is called a stationary point (or endpoint) of S and T if $Tz = \{Sz\}$ (see [10]).

If S is an identity mapping on X, the point $z \in X$ is called a stationary point of T if $Tz = \{z\}$. Note that every stationary point of T is a fixed point of T, but not conversely.

The existence of stationary points (or endpoints) of multi-valued mappings have been studied by several authors (cf. [2–6, 10, 11, 13, 16–19] and others). The purpose of this paper is to obtain some fixed and stationary point theorems in metric spaces. Specifically, in Section 2, we obtain some new fixed and stationary point theorems for certain classes of mappings without using the completeness of the space and continuity conditions. These mappings satisfy the *property* (E.A) introduced and studied by Aamri and Moutawakil [1] for the first time. It is interesting to note that the above property presents a nice generalization of non-compatible mappings. Section 3 deals with approximate endpoint property. In this section we obtain results on stationary points for generalize hybrid pair of single-valued and multi-valued mappings. Results obtained in this section extend and generalize certain results of [3], [13] and others.

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2. THE PROPERTY (E.A)

Throughout this section we shall use the following notations.

(1) Φ a class of the functions: $\varphi\colon [0,\infty) \to [0,\infty)$ satisfying:
   (a) φ is continuous and monotone nondecreasing,
   (b) $\varphi(t) = 0 \Leftrightarrow t = 0$.

(2) $M_T(x,y) := \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$.

Aamri and Moutawakil [1] introduced the following interesting notion of the property (E.A).

Definition 2.1. Let X be a metric space. Two mappings $S, T\colon X \to X$ satisfy the property (E.A) if there exits a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z, \quad \text{for some } z \in X.$$

Kamran [12], extended the above notion to a hybrid pair of mappings as follows:

Definition 2.2. Let X be a metric space. Two mappings $S\colon X \to X$ and $T\colon X \to CL(X)$ satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = z \in A = \lim_{n\to\infty} Tx_n$$

for some $z \in X$ and $A \in CL(X)$.

When S is an identity mapping on X, we obtain the corresponding definition for a (single) mapping satisfying the property (E.A) (see [14]).

Example 2.1. [1]. Let $X = [0,\infty)$ endowed with the usual metric. Define $S, T\colon X \to X$ by $Sx = \dfrac{x}{4}$ and $Tx = \dfrac{3x}{4}$ for all $x \in X$. Consider the sequence $x_n = \dfrac{1}{n}$, $n \in \mathbb{N}$. Clearly $\lim\limits_{n\to\infty} Sx_n = \lim\limits_{n\to\infty} Tx_n = 0$, and S and T satisfy the property (E.A).

There are mappings which do not satisfy the property (E.A).

Example 2.2. [1]. Let $X = [2,\infty)$ endowed with the usual metric. Define $S, T\colon X \to X$ by $Sx = x+1$ and $Tx = 2x+1$, for all $x \in X$. Suppose S and T satisfy the property (E.A), then there exists a sequence $\{x_n\}$ in X satisfying $\lim\limits_{n\to\infty} Sx_n = \lim\limits_{n\to\infty} Tx_n = t$, for some $t \in X$. Therefore $\lim\limits_{n\to\infty} x_n = t-1$ and $\lim\limits_{n\to\infty} x_n = \dfrac{t-1}{2}$. Then $t = 1$, which is a contradiction since $1 \notin X$. Hence S and T do not satisfy the property (E.A).

Example 2.3. Let $X = [0,1]$ endowed with the usual metric. Define $S\colon X \to X$ and $T\colon X \to CL(X)$ by $Sx = \dfrac{x}{2}$ and $Tx = [0,x]$. Consider the sequence $x_n = \dfrac{1}{n}$, $n \in \mathbb{N}$. Clearly,

$$\lim_{n\to\infty} Sx_n = 0 \in \{0\} = \lim_{n\to\infty} Tx_n.$$

Therefore, S and T satisfy the property (E.A).

Definition 2.3. [2, 9] Let X be a metric space and $T\colon X \to X$. The mapping T is said to be (ψ,φ)-*weak contraction* if

$$\psi(d(Tx,Ty)) \le \psi(d(x,y)) - \varphi(d(x,y)) \tag{2.1}$$

for all $x, y \in X$, where $\psi, \varphi \in \Phi$.

We extend Definition 2.3 and introduce the notion of a quasi weak contraction as follows:

Definition 2.4. Let X be a metric space and $T\colon X \to X$. The mapping T will be called a *quasi weak contraction* if

$$\psi(d(Tx,Ty)) \le \psi(M_T(x,y)) - \varphi(M_T(x,y)) \tag{2.2}$$

for all $x, y \in X$, where $\psi, \varphi \in \Phi$.

When $\psi(t) = t$ and $\varphi(t) = (1-k)t$ with $k \in (0,1)$, in the Definition 2.4, we recover the well known quasi-contraction due to Ćirić [8].

We remark that Ćirić's quasi-contraction [8] is considered as the most general among contractions listed in Rhoades [15].

In case T is a multi-valued mapping, we have the following definition.

Definition 2.5. Let X be a metric space and $T\colon X \to CL(X)$. The mapping T will be called a *multi-valued quasi weak contraction* if

$$\psi(H(Tx,Ty)) \le \psi(M_T(x,y)) - \varphi(M_T(x,y)) \tag{2.3}$$

for all $x, y \in X$, where $\psi, \varphi \in \Phi$.

Now we present our first result.

Theorem 2.1. *Let X be a metric space and $T\colon X \to CL(X)$ a multi-valued quasi weak contraction satisfying the property* (E.A)*. Then T has a unique stationary point in X.*

Proof. Since T satisfies the property (E.A), there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} x_n = z \in A = \lim_{n\to\infty} Tx_n$$

for some $z \in X$ and $A \in CL(X)$. By (2.3), we get

$$\psi(H(Tz,Tx_n)) \le \psi(M_T(z,x_n)) - \varphi(M_T(z,x_n)). \tag{2.4}$$

Notice that

$$\begin{aligned}\lim_{n\to\infty} M_T(z,x_n) &= \lim_{n\to\infty} \max\{d(z,x_n), d(z,Tz), d(x_n,Tx_n), d(z,Tx_n), d(x_n,Tz)\}\\ &= \max\{d(z,z), d(z,Tz), d(z,A), d(z,A), d(z,Tz)\}\\ &= \max\{0, d(z,Tz), 0, 0, d(z,Tz)\}\\ &= d(z,Tz).\end{aligned} \tag{2.5}$$

Since $\psi,\ \varphi \in \Phi$, (2.4) and (2.5) implies

$$\lim_{n\to\infty} \psi(H(Tz,Tx_n)) = \psi(H(Tz,A)) \le \psi(d(z,Tz)) - \varphi(d(z,Tz)),$$

a contradiction, unless $H(Tz,A) = 0$. Thus $Tz = A = \{z\}$.

To prove the uniqueness, we suppose that T has two distinct stationary points u and v in X.

Again notice that

$$\begin{aligned}M_T(u,v) &= \max\{d(u,v), d(u,Tu), d(v,Tv), d(u,Tv), d(v,Tu)\}\\ &= \max\{d(u,v), d(u,u), d(v,v), d(u,v), d(v,u)\}\\ &= d(u,v).\end{aligned} \tag{2.6}$$

Since $\psi,\ \varphi \in \Phi$, by (2.3) and (2.6)

$$\begin{aligned}\psi(d(u,v)) &= \psi(H(Tu,Tv)) \leq \psi(M_T(u,v)) - \varphi(M_T(u,v)) \\ &= \psi(d(u,v)) - \varphi(d(u,v)),\end{aligned}$$

a contradiction, unless $d(u,v)=0$, i.e. $u=v$. □

If T is a single valued mapping on X, we have the following result.

Corollary 2.1. *Let X be a metric space and $T\colon X \to X$ a quasi weak contraction satisfying the property* (E.A)*. Then T has a unique fixed point in X.*

When $\psi(t)=t$ in Theorem 2.1, we have the following corollary.

Corollary 2.2. *Let X be a metric space and $T\colon X \to CL(X)$ a multi-valued mapping satisfying the property* (E.A) *such that*

$$H(Tx,Ty) \leq M_T(x,y) - \varphi(M_T(x,y)) \tag{2.7}$$

for all $x,y \in X$, where $\varphi \in \Phi$. Then T has a unique stationary point.

Corollary 2.3. *Let X be a metric space and $T\colon X \to CL(X)$ a multi-valued (ψ,φ)-weak contraction mapping satisfying the property* (E.A)*. Then T has a unique stationary point.*

Proof. It comes from Theorem 2.1, when $M_T(x,y)=d(x,y)$. □

When $M_T(x,y)=d(x,y)$, $\psi(t)=t$ and $\varphi(t)=0$ in Theorem 2.1, then we have the following nonexpansive type result.

Corollary 2.4. *Let X be a metric space and $T\colon X \to CL(X)$ a multi-valued mapping satisfying the property* (E.A) *such that*

$$H(Tx,Ty) \leq d(x,y) \tag{2.8}$$

for all $x,y \in X$. Then T has a unique stationary point.

Now we present an example to illustrate our results.

Example 2.4. Let $X=[0,1)$ endowed with the usual metric. Define $T\colon X \to CL(X)$ by $Tx=\left[0,\frac{x}{2}\right]$. Let $\psi(t)=2t$ and $\varphi(t)=\frac{t}{2}$.

Consider the sequence $\{x_n\}$ defined by $x_n=\frac{1}{n},\ n\in\mathbb{N}$. Clearly,

$$\lim_{n\to\infty} x_n = 0 \in \{0\} = \lim_{n\to\infty} Tx_n,$$

and T satisfies the property (E.A). For all $x,y \in X$

$$\psi(H(Tx,Ty)) = |x-y| \leq 2|x-y| - \frac{|x-y|}{2}.$$

Therefore T satisfies all the hypotheses of Corollary 2.3 and $T0=\{0\}$. It is interesting to note that X is not complete.

3. APPROXIMATE ENDPOINT PROPERTY

Throughout this section we shall use the following notations.

(1) Ψ a class of the functions: $\phi\colon [0,\infty) \to [0,\infty)$, which are upper semicontinuous, $\phi(t) < t$ for each $t > 0$ and

$$\lim_{n\to\infty} \inf(t - \phi(t)) > 0.$$

(2) $m(x,y) := \max\left\{d(x,y), d(x,Tx), d(y,Ty), \dfrac{d(x,Ty)+d(y,Tx)}{2}\right\}$.

(3) $M_{S,T}(x,y) := \max\left\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), \dfrac{d(Sx,Ty)+d(Sy,Tx)}{2}\right\}$.

Let $S\colon X \to X$ and $T\colon X \to CL(X)$. We say that S and T have approximate endpoint property if

$$\inf_{z\in X} \sup_{y\in Tz} d(Sz,y) = 0.$$

Furthermore, T has approximate endpoint property if

$$\inf_{z\in X} \sup_{y\in Tz} d(z,y) = 0.$$

Example 3.1. Let us consider $X = [0,1]$ endowed with the usual metric. Let $S\colon X \to X$ and $T\colon X \to CL(X)$ be given by

$$Sx = x \text{ for all } x \in X; \text{ and } \ Tx = \begin{cases} \left\{\dfrac{1}{2}\right\}, & \text{if } x = 0 \\ [0,x], & \text{if } x \neq 0. \end{cases}$$

Then it can be easily seen that

$$\inf_{z\in X} \sup_{y\in Tz} d(Sz,y) = 0.$$

So S and T have approximate endpoint property. But S and T have no stationary point.

Recently, extending Boyd and Wong contraction [7] for a multi-valued mapping, Amini-Harandi [3] obtained the following stationary point theorems.

Theorem 3.1. *Let X be a complete metric space and $T\colon X \to CB(X)$ such that*

$$H(Tx,Ty) \leq \phi(d(x,y)) \text{ for all } x,y \in X. \tag{3.1}$$

Then T has a unique endpoint if and only if T has approximate endpoint property.

Moradi and Khojasteh [13] extended the above theorem to a multi-valued generalized weak contraction as follows:

Theorem 3.2. *Let X be a complete metric space and $T\colon X \to CB(X)$ such that*

$$H(Tx,Ty) \leq \phi(m(x,y)) \text{ for all } x,y \in X. \tag{3.2}$$

Then T has a unique endpoint if and only if T has approximate endpoint property.

The following result is an extension of Theorems 3.1 and 3.2.

Theorem 3.3. *Let X be a metric space, $S\colon X \to X$ and $T\colon X \to CB(X)$ such that*
(A1) $TX \subseteq SX$;
(A2) $H(Tx,Ty) \le \phi(M_{S,T}(x,y))$ *for all* $x, y \in X$;
(A3) SX *or* TX *is a complete subspace of* X.
Then S and T have a unique endpoint if and only if S and T have approximate endpoint property.

Proof. It is obvious that if S and T have an endpoint then they have approximate endpoint property. Conversely, suppose that S and T have approximate endpoint property. Then there exists a sequence $\{x_n\} \in X$ such that $\lim\limits_{n\to\infty} H(Sx_n, Tx_n) = 0$. We shall show that the sequence $\{Sx_n\}$ is Cauchy. For all $m, n \in \mathbb{N}$

$$\begin{aligned}
M_{S,T}(x_n,x_m) &= \max\left\{d(Sx_n,Sx_m), d(Sx_n,Tx_n), d(Sx_m,Tx_m), \frac{d(Sx_n,Tx_m)+d(Sx_m,Tx_n)}{2}\right\}\\
&\le \max\left\{d(Sx_n,Sx_m), H(Sx_n,Tx_n), H(Sx_m,Tx_m), \frac{H(Sx_n,Tx_m)+H(Sx_m,Tx_n)}{2}\right\}\\
&\le d(Sx_n,Sx_m) + H(Sx_n,Tx_n) + H(Sx_m,Tx_m)\\
&= d(Sx_n,Sx_m) - H(Sx_n,Tx_n) - H(Sx_m,Tx_m) + 2H(Sx_n,Tx_n) + 2H(Sx_m,Tx_m)\\
&\le H(Tx_n,Tx_m) + 2H(Sx_n,Tx_n) + 2H(Sx_m,Tx_m)\\
&\le \phi(M_{S,T}(x_n,x_m)) + 2H(Sx_n,Tx_n) + 2H(Sx_m,Tx_m).
\end{aligned}$$

Therefore for all $m, n \in \mathbb{N}$, we have

$$M_{S,T}(x_n,x_m) - \phi(M_{S,T}(x_n,x_m)) \le 2[H(Sx_n,Tx_n) + H(Sx_m,Tx_m)]. \tag{3.3}$$

Since ϕ is upper semicontinuous, $\phi(t) < t$ for each $t > 0$ and $\lim\limits_{n\to\infty} \inf(t - \phi(t)) > 0$, by (3.3) we conclude that

$$\limsup_{m,n} M_{S,T}(x_n,x_m) = 0,$$

and $\{Sx_n\}$ is a Cauchy sequence.

Suppose SX is a complete subspace of X. Then $\{Sx_n\}$ being contained in SX has a limit in SX. Call it z. Let $Su = z$ for some $u \in X$. Now we have

$$\begin{aligned}
M_{S,T}(x_n,u) &= \max\left\{d(Sx_n,Su), d(Sx_n,Tx_n), d(Su,Tu), \frac{d(Sx_n,Tu)+d(Su,Tx_n)}{2}\right\}\\
&\le \max\left\{d(Sx_n,Su), H(Sx_n,Tx_n), H(Su,Tu), \frac{H(Sx_n,Tu)+H(Su,Tx_n)}{2}\right\}\\
&\le d(Sx_n,Su) + H(Sx_n,Tx_n) + H(Su,Tu),
\end{aligned}$$

and

$$\lim_{n\to\infty} M_{S,T}(x_n,u) \le H(Su,Tu).$$

Since ϕ is upper semicontinuous

$$\lim_{n\to\infty} \sup \phi(M_{S,T}(x_n,u)) \le \phi(H(Su,Tu)). \tag{3.4}$$

By the triangle inequality and using (A2), we have

$$\begin{aligned}
H(Sx_n,Tu) &\le H(Sx_n,Tx_n) + H(Tx_n,Tu)\\
&\le H(Sx_n,Tx_n) + \phi(M_{S,T}(x_n,u)).
\end{aligned}$$

Making $n \to \infty$ and using (3.4), we get

$$H(Su,Tu) \le \phi(H(Su,Tu)),$$

and $\phi(t) < t$ for all $t > 0$, implies that $H(Su, Tu) = 0$. It follows that $Tu = \{Su\}$ and u is an endpoint of S and T.

In case TX is a complete subspace of X, the condition $TX \subseteq SX$ implies that the sequence $\{Sx_n\}$ converges in SX and the previous argument works. The uniqueness of endpoint follows easily. □

Corollary 3.1. *Theorem* 3.2.

Proof. It comes from Theorem 3.3 when S is an identity mapping on X. □

Corollary 3.2. *Let X be a metric space, $S\colon X \to X$ and $T\colon X \to CB(X)$ such that*
(C1) *$TX \subseteq SX$;*
(C2) *$H(Tx, Ty) \leq q\ M_{S,T}(x, y)$ for all $x, y \in X$, where $0 < q < 1$;*
(C3) *SX or TX is a complete subspace of X.*
Then S and T have a unique endpoint if and only if S and T have approximate endpoint property.

Proof. It comes from Theorem 3.3 when $\phi(t) = qt$. □

Corollary 3.3. *Theorem* 3.1.

Proof. It comes from Corollary 3.1 when $m(x, y) = d(x, y)$. □

Following example shows the generality of our Theorem 3.3.

Example 3.2. Let $X = [0, \infty)$ endowed with the usual metric. Let $S\colon X \to X$ and $T\colon X \to CB(X)$ be given by

$$Sx = 2x \text{ for all } x \in X; \text{ and } \ Tx = \begin{cases} \{0\}, & \text{if } x = 0 \\ [0, x], & \text{if } x \neq 0. \end{cases}$$

Consider $\phi(t) = \dfrac{3}{4}t$ for $t > 0$. Then for $x, y > 0$

$$H(Tx, Ty) = |x - y| > \frac{3}{4}|x - y|,$$

and the condition (3.1) (of Theorem 3.1) and condition (3.2) (of Theorem 3.2) are not satisfied. However, the mappings S and T satisfy all the conditions of Theorem 3.3 and $S0 \in T0 = \{0\}$, showing that 0 is the unique endpoint of S and T.

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