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FIXED POINT THEOREMS FOR OCCASIONALLY WEAKLY COMPATIBLE EXPANSIVE MAPPINGS

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Abstract. Gerald Jungck and the second author wrote a paper proving a very general theorem for occasionally weakly compatible (owc) mappings. In this paper we establish a general result for expansive owc mappings.

1. INTRODUCTION

The research literature on fixed point theory contains over a thousand papers, each one devoted to proving the existence (and, in most cases, uniqueness) of a fixed point for a map (or a pair of maps) satisfying some contractive type condition. In 1976 Jungck [1] proved a generalization of the Banach contraction principle by introducing a second function on the right hand side of the inequality. The contractive condition takes the form $d(fx, fy) \leq kd(gx, gy)$, where f and g are selfmaps of a complete metric space X and k is a constant satisfying $0 \leq k < 1$. In order to obtain a successful iteration scheme it was assumed that $f(X) \subseteq g(X)$. He also assumed that f and g commute. This provided a stimulus for a number of such papers dealing with two, three, or four maps, assuming appropriate range conditions, and generalizing commutativity to weakly commuting, etc. In 2006 Jungck and the second author [2] proved a general theorem of this type, which contains many of the theorems in the literature as special cases. The purpose of this paper is to prove a similar general result for expansive mappings.

2. MAIN RESULTS

Let X be a set. A symmetric on X is a mapping $r\colon X \times X \to [0, \infty)$ such that

$$r(x, y) = 0 \quad \text{iff} \quad x = y, \quad \text{and} \quad r(x, y) = r(y, x) \quad \text{for} \quad x, y \in X.$$

Let f and g be selfmaps of X. A point x in X is called a coincidence point of f and g iff $fx = gx$. We shall call $w = fx = gx$ a point of coincidence of f and g. Two selfmaps f and g are called occasionally weakly compatible (owc) iff there is a coincidence point of f and g at which f and g commute.

Theorem 2.1. *Let r be a symmetric on a set X. Suppose that f, g, S, and T are selfmaps of X, with (f, S) and (g, T) owc, satisfying*

$$r(fx, gy) \geq \psi(r(Sx, Ty), r(Sx, fx), r(Ty, gy)), \tag{2.1}$$

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for all $x, y \in X$, where $\psi\colon \mathbb{R}_+^3 \to \mathbb{R}_+$ satisfies $\psi(\alpha, 0, 0) > \alpha$ for each $\alpha > 0$.

Then f, g, S, and T have a unique common fixed point in X.

Proof. By hypothesis there exist points $u, v \in X$ such that $w := fu = Su$ and $gv = Tv$. Suppose that $fu \neq gv$. Substituting in (2.1),

$$\begin{aligned} r(fu, gv) &\geq \psi(r(Su, Tv), r(Su, fu), r(gv, Tv)) \\ &= \psi(r(fu, gv), 0, 0) > r(fu, gv), \end{aligned}$$

a contradiction. Therefore $fu = gv$.

Suppose that there exists another point $z \in X$ such that $fz = Sz$, with $w \neq z$. Then, using (2.1),

$$\begin{aligned} r(fz, gv) &\geq \psi(r(Sz, Tv), r(Sz, fz), r(Tv, gv)) \\ &= \psi(r(fz, gv), 0, 0) > r(fz, gv), \end{aligned}$$

a contradiction. Therefore the coincidence point is unique. From Lemma 1 of [2], w is the unique fixed point of f and g. By symmetry, there is a unique point $p \in X$ such that $p = gz = Tz$.

Suppose that $w \neq p$. Again using (2.1),

$$\begin{aligned} r(w, p) = r(fu, gz) &\geq \psi(r(Su, Tz), r(Su, fu), r(gz, Tz)) \\ &= \psi(r(p, q), 0, 0) > r(p, q), \end{aligned}$$

a contradiction. Therefore p is a common fixed point of f, g, S, and T.

Suppose that there exists another common fixed point q with $p \neq q$. Again using (2.1), we get

$$\begin{aligned} r(p, q) = r(fp, gq) &\geq \psi(r(Sp, Tq), r(Sp, fp), r(Tq, gq)) \\ &= \psi(r(p, q), 0, 0) > r(p, q), \end{aligned}$$

a contradiction Therefore the common fixed point is unique. □

Corollary 2.1. *Let r be a symmetric on a set X. Suppose that f, g, S, and T are selfmaps of X, with (f, S) and (g, T) owc, satisfying*

$$r(fx, gy) \geq ar(Sx, Ty) + br(Sx, fx) + cr(Ty, gy)$$

for all $x, y \in X$, where $a, c > 1$, and $b \in \mathbb{R}$ such that $a + b + c > 1$. Then f, g, S, and T have a unique common fixed point in X.

Proof. Define ψ by $\psi(t_1, t_2, t_3) = at_1 + bt_2 + ct_3$. Then apply Theorem 2.1. □

Corollary 2.2. *Let r be a symmetric on a set X. Suppose that f, g, S, and T are selfmaps of X, with (f, S) and (g, T) owc, satisfying*

$$r(fx, gy) \geq kr(Sx, Ty) \tag{2.2}$$

for all $x, y \in X$, where $k > 1$. Then f, g, S, and T have a unique common fixed point.

Proof. Define ψ by $\psi(t_1, t_2, t_3) = kt_1$. Now apply Theorem 2.1. □

The authors in [6] define semi-compatability of two maps. This definition is the same as owc.

Corollary 2.3. ([6], Theorem 4.1) *Let A, B, S, T be self-maps of an arbitrary set M such that each of the pairs (A, S) and (B, T) are semi-compatible mappings and satisfy*

$$d(Sx, Ty) \geq \psi(d(Ax, By), d(Ax, Sx), d(By, Ty)),$$

and there exist u, v and z in M such that $Au = Su = Bv = Tv = z$. Also assume that

$$A(M) \subseteq T(M), \quad \text{and} \quad B(M) \subseteq S(M),$$

and that $S(M)$ is d-complete.

Then A, B, S, T have a unique common fixed point in M.

Proof. Simply observe that the authors have assumed that the maps are pairwise owc, along with other conditions. Thus the result follows from Theorem 2.1. □

Two maps S and T are said to be compatible of type (A) if $\lim_n d(TSx_n, S^2x_n) = 0$ and $\lim_n d(STx_n, T^2x_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_n Sx_n = \lim_n Tx_n = z$ for some $z \in X$.

A map $\psi\colon \mathbb{R}_+^3 \to \mathbb{R}_+$ is said to satisfy property (h) if there exist an $h \geq 1$ such that, for every $u, v \in \mathbb{R}_+$, with $u \geq \psi(v, u, v)$ or $u \geq \psi(v, v, u)$ we have $u \geq hv$.

Corollary 2.4. ([5], Theorem 1) *Let A, B, S, and T be selfmaps of a complete metric space (X, d) satisfying:*

1) *A and B are surjective;*
2) *one of A, B, S, T is continuous;*
3) *A and S as well as B and T are compatible of type (A);*
4) *inequality*

$$d(Ax, By) \geq \psi(d(Sx, Ty), d(Ax, Sx), d(By, Ty))$$

holds for all $x, y \in X$, where ψ satisfies property (h) with $h > 1$.

If $\psi(\alpha, 0, 0) > \alpha$ for each $\alpha > 0$, and ψ is continuous, then A, B, S, T have a unique common fixed point.

Proof. The inequality is the same as that of Theorem 2.1. In the course of the proof of Theorem 1 of [5] it is shown that (A, S) and (B, T) are owc. The result then follows from Theorem 2.1. □

A map $\psi\colon \mathbb{R}_+^3 \to \mathbb{R}_+$ is said to satisfy Property (B) if, for every $u, v \in \mathbb{R}_+$ such that $u \geq \psi(v, u, v)$ we have $u \geq hv$, where $\psi(1, 1, 1) = h \geq 1$. The map ψ is said to satisfy Property (B^*) if, for every $u, v \in \mathbb{R}_+$ such that $u \geq \psi(v, u, u)$, we have $u \geq hv$, where $\psi(1, 1, 1) = h \geq 1$. A map ψ is said to satisfy Property (u) if $\psi(u, 0, 0) > u$ for each $u > 0$.

Corollary 2.5. ([3], Theorem 2) *Let A, B, S, and T be selfmaps of a complete metric space (X, d) satisfying:*

1) *A and B are surjective;*
2) *one of A, B, S, T is continuous;*
3) *A and S as well as B and T are compatible;*
4) *inequality*

$$d(Ax, By) \geq \psi(d(Sx, Ty), d(Ax, Sx), d(By, Ty))$$

holds for all $x, y \in X$, where ψ satisfies Properties (B) and (B^*) with $h > 1$.

If Property (u) holds and ψ is continuous, then A, B, S and T have a unique common fixed point.

A map $\phi\colon \mathbb{R}_+^5 \to \mathbb{R}_+$ is said to satisfy Property (u) if $\phi(u,0,0,u,u) > u$ for each $u > 0$.

Theorem 2.2. *Let f, g, S and T be selfmaps of a complete metric space (X,d), such that (f,S) and (g,T) are owc, satisfying*

$$d(Sx,Ty) \geq \phi(d(fx,gy), d(Sx,fx), d(Ty,gy), d(Sx,gy), d(Ty,fx)) \tag{2.3}$$

for all $x,y \in X$, where ϕ satisfies Property (U). Then f,g,S, and T have a unique common fixed point.

Proof. By hypothesis there exist points u and v such that $w := fu = Su$ and $gv = Tv$. Suppose that $fu \neq gv$. Then, from (2.2),

$$\begin{aligned} d(fu,gv) &\geq \phi(d(Su,Tv), d(Su,fu), d(Tv,gv), d(Su,gv), d(Tv,gu)) \\ &= \phi(d(fu,gv), 0, 0, d(fu,gv), d(fu,gv)) > d(fu,gv), \end{aligned}$$

a contradiction. Therefore $fu = gv$. Suppose that there exists another point z such that $fz = gv$, with $fz \neq gv$. Then, using (2.2),

$$\begin{aligned} d(fz,gv) &\geq \phi(d(Sz,Tv), d(Sz,fz), d(Tv,gv), d(Sz,gv), d(Tv,fz)) \\ &= \phi(d(fz,gv), 0, 0, d(fz,gv), d(fz,gv)) > d(fz,gv), \end{aligned}$$

a contradiction. From Lemma 1 of [2], f and S have a unique common fixed point. Similarly, g and T have a unique common fixed point. □

Let $\mathcal{H}$ denote the set of real continuous functions $g(x_1,\ldots,x_5)\colon \mathbb{R}_+^5 \to \mathbb{R}_+$ satisfying the following conditions:

H_1: g is decreasing in variables x_4 and x_5,

H_2: there exists an $h > 1$ such that, for every $, v \geq 0$ with

$$H_3 : u \geq g(v,v,u,0,u+v) \quad \text{or} \quad H_b : u \geq g(v,u,v,u+v,0),$$

we have $u \geq hv$.

Corollary 2.6. ([4], Theorem 3) *Let (X,d) be a metric space, A, B, S and T selfmaps of X satisfying:*

1) *A and B are surjective;*

2) *one of A,B,S,T is continuous;*

3) *A and S as well as B and T are compatible of type (A).*

If inequality (2.4) holds for all $x,y \in X$, where $g \in \mathcal{H}$, and where

$$d(Ax,By) \geq g(d(Sx,Ty), d(Ax,Sx), d(Ty,By), d(Ax,Ty), d(By,Sx)), \tag{2.4}$$

for all $x,y \in X$, where g satisfies Property (u), then A, B, S, T have a unique common fixed point.

Proof. Inequality (2.4) is the same as inequality (2.3). In the course of the proof of Theorem 3 of [4] it is shown that (S,A) and (T,B) are owc. Therefore the conclusion follows from Theorem 2.2. □

3. CONCLUDING REMARKS

In this paper, we established a general result for expansive owc mappings. The general theorem introduced and proved here clearly eliminates the need to prove any fixed point theorems for specific expansive maps satisfying any of the generalized commutativity conditions such as weakly commuting, compatibility of type (A), etc.

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