J. Adv. Math. Stud.
Vol. **5**(2012), No. 2, 59-67
http://journal.fairpartners.ro

A GENERALIZATION OF BANACH CONTRACTION PRINCIPLE IN ORDERED CONE METRIC SPACES

S. K. MALHOTRA, S. SHUKLA AND R. SEN

Abstract. In this paper we prove fixed point theorems for ordered Ćirić-Prešić type contraction in cone metric spaces without assuming the normality of cone. These results generalize and extend the Banach contraction principle and several known results in ordered cone metric spaces. Some examples are presented to illustrate the cases when new results can be applied while old one can not.

1. INTRODUCTION

K-metric and K-normed spaces were introduced in the mid-20th century (see [2, 16, 24, 26]) by using an ordered Banach space instead of the set of real numbers, as the codomain for a metric. Indeed this idea of replacement of real numbers by an ordered "set" can be seen in [17, 18] (see also references therein). Huang and Zhang [9] re-introduced such spaces under the name of cone metric spaces, defining convergent and Cauchy sequence in terms of interior points of underlying cone. They proved the basic version of the fixed point theorem with the assumption that cone is normal, which were generalized by several authors (see, e.g. [1, 7, 8, 10, 11, 12, 13, 15, 21, 23, 25]). Rezapour and Hamlbarani [23] removed the assumption of normality of cone and generalized the results of Huang and Zhang in non-normal cone metric spaces.

Ordered normed spaces and cones have applications in applied mathematics (see, e.g. [6, 24]). The existence of fixed point in partially ordered sets was investigated by Ran and Reurings [22] and then by Nieto and Lopez [19]. Fixed point results in ordered cone metric spaces were obtained by several authors (see e.g. [3, 4, 14]).

Banach contraction principle is one of the most interesting and useful tool in applied mathematics. Prešić [20] generalized the Banach contraction principle in metric spaces and proved following theorem:

Theorem 1.1. *Let (X, d) be a complete metric space, k a positive integer, $T\colon X^k \to X$ a mapping such that*

$$d(T(x_1, \ldots, x_k), T(x_2, \ldots, x_{k+1})) \leq q_1 d(x_1, x_2) + q_2 d(x_2, x_3) + \cdots + q_k d(x_k, x_{k+1}),$$

for every $x_1, x_2, \ldots, x_{k+1} \in X$, where $q_1, q_2, \ldots, q_k$ are nonnegative constants which satisfy $q_1 + q_2 + \cdots + q_k < 1$. Then there exists a unique point $x \in X$ such that $T(x, x, \ldots, x) = x$. Moreover if $x_1, x_2, \ldots, x_k$ are arbitrary in X and $x_{n+k} = T(x_n, x_{n+1}, \ldots, x_{n+k-1})$, $n \in \mathbb{N}$ then the sequence $\{x_n\}$ is convergent and $\lim x_n = T(\lim x_n, \lim x_n, \ldots, \lim x_n)$.

Received: February 28, 2012. *Revised:* May 25, 2012.
2010 *Mathematics Subject Classification*: 47H10, 54H25.
Key words and phrases: Cone metric space, partial order, fixed point.

©2012 Fair Partners Team for the Promotion of Science & Fair Partners Publishers

Note that for $k = 1$ Prešić theorem turns in to Banach contraction principle. Thus theorem 1.1 is a generalization of the Banach fixed point theorem. Inspired with the results in Theorem 1.1 Ćirić and Prešić [5] proved following theorem:

Theorem 1.2. *Let (X, d) be a complete metric space, k a positive integer and $T\colon X^k \to X$ a mapping satisfying the following contractive type condition:*

$$d(T(x_1, x_2, \ldots, x_k), T(x_2, x_3, \ldots, x_{k+1})) \leq \lambda \max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\},$$

where $\lambda \in (0, 1)$ is constant and $x_1, x_2, \ldots, x_{k+1}$ are arbitrary points in X. Then there exists a point x in X such that $T(x, x, \ldots, x) = x$. Moreover if $x_1, x_2, \ldots, x_k$ are arbitrary points in X and for $n \in \mathbb{N}$, $x_{n+k} = T(x_n, x_{n+1}, \ldots, x_{n+k-1})$, then the sequence $\{x_n\}$ is convergent and $\lim x_n = T(\lim x_n, \lim x_n, \ldots, \lim x_n)$. If in addition we suppose that on diagonal $\triangle \subset X^k$,

$$d(T(u, u, \ldots, u), T(v, v, \ldots, v)) < d(u, v) \text{ holds for } u, v \in X, \text{ with } u \neq v,$$

then x is unique fixed point satisfying $x = T(x, x, \ldots, x)$.

Cone metric version of these Theorems can be seen in [8]. The existence theorems for Prešić type mappings in ordered metric spaces have not been investigated yet. The purpose of this paper is to generalize and extend theorems 1.1 and 1.2 in ordered cone metric spaces. Examples included which illustrate the cases when new results can be applied while old one can not.

2. PRELIMINARIES

We need the following definitions and results, consistent with [6] and [9].

Definition 2.1. [9] Let E be a real Banach space and P be a subset of E. The set P is called a *cone* if:

i) P is closed, nonempty and $P \neq \{\theta\}$, here θ is the zero vector of E;

ii) $a, b \in \mathbb{R},\ a, b \geq 0,\ x, y \in P \Rightarrow ax + by \in P$;

iii) $x \in P$ and $-x \in P \Rightarrow x = \theta$.

Given a cone $P \subset E$, we define a partial ordering "$\preceq$" with respect to P by $x \preceq y$ if and only if $y - x \in P$. We write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$. While $x \ll y$ if and only if $y - x \in P^0$, where P^0 denotes the interior of P.

Let P be a cone in a real Banach space E, then P is called normal, if there exist a constant $K > 0$ such that for all $x, y \in E$,

$$\theta \preceq x \preceq y \text{ implies } \|x\| \leq K\|y\|.$$

The least positive number K satisfying the above inequality is called the normal constant of P.

Definition 2.2. [9] Let X be a nonempty set, E be a real Banach space. Suppose that the mapping $d\colon X \times X \to E$ satisfies:

(1) $\theta \preceq d(x, y)$, for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, y) \preceq d(x, z) + d(y, z)$, for all $x, y, z \in X$.

Then P is called a *cone metric* on X, and (X, d) is called a cone metric space. If the underlying cone is normal then (X, d) is called a normal cone metric space.

In the following we always suppose that E is a real Banach space, P is a solid cone in E i.e. $P^0 \neq \phi$ and " $\preceq$ " is partial ordering with respect to P.

The concept of cone metric space is more general then that of a metric space, because each metric space is a cone metric space with $E = \mathbb{R}$ and $P = [0, +\infty)$.

For examples and basic properties of normal and non-normal cones and cone metric spaces we refer [9] and [23].

The following remark will be useful in sequel.

Remark 2.1. [13] Let P be a cone in a real Banach space E, and $a, b, c \in P$, then:

a) If $a \preceq b$ and $b \ll c$ then $a \ll c$.
b) If $a \ll b$ and $b \ll c$ then $a \ll c$.
c) If $\theta \preceq u \ll c$ for every $c \in P^0$ then $u = \theta$.
d) If $c \in P^0$, $a_n \in E$ and $a_n \to \theta$ then there exist $n_0 \in \mathbb{N}$ such that, for all $n > n_0$ we have $a_n \ll c$.
e) If $\theta \preceq a_n \preceq b_n$ for each n and $a_n \to a$, $b_n \to b$ then $a \preceq b$.
f) If $a \preceq \lambda a$ where $0 \leq \lambda < 1$ then $a = \theta$.

Definition 2.3. Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$.

(1) If for every $c \in E$ with $\theta \ll c$ (or equivalently $c \in P^0$) there is $n_0 \in \mathbb{N}$ such that, $d(x_n, x) \ll c$, for all $n > n_0$, then the sequence $\{x_n\}$ is said to be *convergent* and converges to x. We denote this by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
(2) If for every $c \in E$, with $\theta \ll c$ there is $n_0 \in \mathbb{N}$ such that, $d(x_n, x_m) \ll c$, for all $n, m \geq n_0$, then the sequence $\{x_n\}$ is called a *Cauchy sequence* in X.
(3) (X, d) is said to be a *complete* cone metric space, if every Cauchy sequence in X is convergent in X.

Definition 2.4. Let (X, d) be a cone metric space, k a positive integer and $f\colon X^k \to X$ be a mapping. If $f(x, x, \ldots, x) = x$, then $x \in X$ is called a *fixed point* of f.

Definition 2.5. Let a nonempty set X is equipped with a partial order "$\sqsubseteq$" such that (X, d) is a cone metric space, then $(X, \sqsubseteq, d)$ is called an *ordered cone metric space*. A sequence $\{x_n\}$ in X is said to be *nondecreasing* with respect to "$\sqsubseteq$" if $x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq \cdots$. Let k be a positive integer and $f\colon X^k \to X$ be a mapping, then f is said to be nondecreasing with respect to "$\sqsubseteq$" if for any finite nondecreasing sequence $\{x_n\}_{n=1}^{k+1}$ we have $f(x_1, x_2, \ldots, x_k) \sqsubseteq f(x_2, x_3, \ldots, x_{k+1})$.

Remark 2.2. For $k = 1$ above definitions reduce to usual definitions of fixed point and nondecreasing mapping in cone metric space.

Let $(X, \sqsubseteq, d)$ be an ordered cone metric space, k a positive integer and $f\colon X^k \to X$ a mapping.

f is said to be an ordered Prešić type contraction if

$$d(f(x_1, \ldots, x_k), f(x_2, \ldots, x_{k+1})) \preceq q_1 d(x_1, x_2) + q_2 d(x_2, x_3) + \cdots + q_k d(x_k, x_{k+1})$$

for all $x_1, x_2, \ldots, x_{k+1} \in X$ with $x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_{k+1}$ and nonnegative constants $q_1, q_2, \ldots, q_k$ such that $q_1 + q_2 + \cdots + q_k < 1$.

f is said to be an ordered Ćirić-Prešić type contraction if

$$d(f(x_1, \ldots, x_k), f(x_2, \ldots, x_{k+1})) \preceq \lambda \max\{d(x_1, x_2), d(x_2, x_3), \ldots, d(x_k, x_{k+1})\}$$

for all $x_1, x_2, \ldots, x_{k+1} \in X$ with $x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_{k+1}$ and $\lambda \in (0, 1)$, provided the "max" exists in E.

Consider a function $\phi\colon E^k \to E$ such that:

(1) $\phi(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_k) \preceq \phi(x_1, x_2, \ldots, x_{i+1}, y_i, x_{i+1}, \ldots, x_k)$ whenever $x_i \preceq y_i$ for any i with $1 \leq i \leq k$;
(2) $\phi(x, x, \ldots, x) \preceq x$ for all $x \in E$.

We write $\phi \in \Phi$ if ϕ satisfies above properties.

f is said to be an ordered ϕ-Ćirić-Prešić type contraction if

$$d(f(x_1, x_2, \ldots, x_k), f(x_2, x_3, \ldots, x_{k+1})) \preceq \lambda\phi(d(x_1, x_2), d(x_2, x_3), \ldots, d(x_k, x_{k+1})) \quad (2.1)$$

for all $x_1, x_2, \ldots, x_{k+1} \in X$ with $x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_{k+1}$, $\phi \in \Phi$ and $\lambda \in (0, 1)$.

3. MAIN RESULTS

Theorem 3.1. *Let $(X, \sqsubseteq, d)$ be an ordered complete cone metric space with solid cone P. Let k be a positive integer, $f\colon X^k \to X$ be a nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following conditions hold:*

i) *f is ordered ϕ-Ćirić-Prešić type contraction i.e. satisfies (2.1);*
ii) *there exist $x_1, x_2, \ldots, x_k \in X$ such that $x_k \sqsubseteq f(x_1, x_2, \ldots, x_k)$, $x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_k$ and $\mu = \max\left\{\frac{d(x_1, x_2)}{\delta}, \frac{d(x_2, x_3)}{\delta^2}, \ldots, \frac{d(x_k, f(x_1, x_2, \ldots, x_k))}{\delta^k}\right\}$ exists in E where $\delta = \lambda^{\frac{1}{k}}$;*
iii) *if a nondecreasing sequence $\{x_n\}$ converges to $x \in X$, then $x_n \sqsubseteq x$ for all $n \in \mathbb{N}$.*

Then f has a fixed point $u \in X$.

Proof. Starting with given $x_1, x_2, \ldots, x_k \in X$, we construct a nondecreasing sequence $\{x_n\}$ as follows: since $x_1, x_2, \ldots, x_k \in X$ satisfy $x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_k$ and $x_k \sqsubseteq f(x_1, x_2, \ldots, x_k)$, let $x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1})$, $n = 1, 2, \ldots$, so $x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_k \sqsubseteq x_{k+1}$ and f is nondecreasing with respect to " $\sqsubseteq$ " it implies that $f(x_1, x_2, \ldots, x_k) \sqsubseteq f(x_2, x_3, \ldots, x_{k+1})$ i.e. $x_{k+1} \sqsubseteq x_{k+2}$. Continuing this procedure, we obtain

$$x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_k \sqsubseteq x_{k+1} \sqsubseteq \cdots .$$

Thus $\{x_n\}$ is a nondecreasing sequence with respect to " $\sqsubseteq$ ".

By the method of mathematical induction we shall prove that

$$d_n \preceq \mu\delta^n, \text{ for all } n \in \mathbb{N}. \quad (3.1)$$

where $d_n = d(x_n, x_{n+1})$.

By definition of μ it is clear that (3.1) is true for $n = 1, 2, \ldots, k$. Let the k inequalities $d_n \preceq \mu\delta^n, d_{n+1} \preceq \mu\delta^{n+1}, \ldots, d_{n+k-1} \preceq \mu\delta^{n+k-1}$ be the induction hypothesis. Then since f is ϕ-Ćirić-Prešić type contraction, $x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_{k+1}$ and $0 < \delta < 1$ we obtain

$$\begin{aligned}
d_{n+k} &= d(x_{n+k}, x_{n+k+1}) \\
&= d(f(x_n, x_{n+1}, \ldots, x_{n+k-1}), f(x_{n+1}, x_{n+2}, \ldots, x_{n+k})) \\
&\preceq \lambda\phi(d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \ldots, d(x_{n+k-1}, x_{n+k})) \\
&= \lambda\phi(d_n, d_{n+1}, \ldots, d_{n+k-1}) \\
&\preceq \lambda\phi(\mu\delta^n, \mu\delta^{n+1}, \ldots, \mu\delta^{n+k-1}) \\
&\preceq \lambda\phi(\mu\delta^n, \mu\delta^n, \ldots, \mu\delta^n) \\
&\preceq \lambda\mu\delta^n \\
&= \mu\delta^{n+k} \quad (\text{as } \delta = \lambda^{\frac{1}{k}}).
\end{aligned}$$

Thus inductive proof of (3.1) is complete.

Now we shall show that $\{x_n\}$ is Cauchy sequence in X. Let $n, m \in \mathbb{N}$, then using (3.1) we obtain

$$\begin{aligned} d(x_n, x_{n+m}) &\preceq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+m-1}, x_{n+m}) \\ &= d_n + d_{n+1} + \cdots + d_{n+m-1} \\ &\preceq \mu\delta^n + \mu\delta^{n+1} + \cdots + \mu\delta^{n+m-1} \\ &\preceq \mu(1 + \delta + \delta^2 + \cdots)\delta^n \\ &= \frac{\mu\delta^n}{1-\delta}. \end{aligned} \tag{3.2}$$

Since $0 < \delta < 1$ it follows that $\dfrac{\mu\delta^n}{1-\delta} \to \theta$ as $n \to \infty$, therefore from d) of Remark 2.1, for every $c \in P^0$ there exists $n_0 \in \mathbb{N}$ such that $\dfrac{\mu\delta^n}{1-\delta} \ll c$ for all $n > n_0$, and by a) of Remark 2.1 and (3.2) we obtain, $d(x_n, x_{n+m}) \ll c$ for all $n > n_0$. Therefore $\{x_n\}$ is a nondecreasing Cauchy sequence in X, and since X is complete, there exists $u \in X$ such that $\lim\limits_{n\to\infty} x_n = u$. Again by iii) $x_n \sqsubseteq u$ for all $n \in \mathbb{N}$ and f is ordered ϕ-Ćirić-Prešić type contraction therefore

$$\begin{aligned} d(u, f(u,u,\ldots,u)) &\preceq d(u, x_{n+k}) + d(x_{n+k}, f(u,u,\ldots,u)) \\ &= d(f(x_n, x_{n+1}, \ldots, x_{n+k-1}), f(u,u,\ldots,u)) + d(u, x_{n+k}) \\ &\preceq d(f(x_n, x_{n+1}, \ldots, x_{n+k-1}), f(x_{n+1}, x_{n+2}, \ldots, x_{n+k-1}, u)) \\ &\quad + d(f(x_{n+1}, x_{n+2}, \ldots, x_{n+k-1}, u), f(x_{n+2}, x_{n+3}, \ldots, x_{n+k-1}, u, u)) \\ &\quad + \cdots + d(f(x_{n+k-1}, u, u, \ldots, u), f(u,u,\ldots,u)) + d(u, x_{n+k}) \end{aligned}$$

i.e.

$$\begin{aligned} d(u, f(u,u,\ldots,u)) \preceq\ & d(x_{n+k}, u) + \lambda\phi(d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \ldots, d(x_{n+k-1}, u)) \\ & + \lambda\phi(d(x_{n+1}, x_{n+2}), d(x_{n+2}, x_{n+3}), \ldots, d(x_{n+k-1}, u), d(u,u)) \\ & + \cdots + \lambda\phi(d(x_{n+k-1}, u), d(u,u), \ldots, d(u,u)). \end{aligned} \tag{3.3}$$

Using the fact that $\lim\limits_{n\to\infty} x_n = u$ and (3.1), for every $c \in P^0$ there exists $n_1 \in \mathbb{N}$ such that $d(x_n, u) \ll \dfrac{c}{k+1}, d(x_n, x_{n+1}) \ll \dfrac{c}{k+1}$ for all $n > n_1$ also, $\theta = d(u,u) \ll \dfrac{c}{k+1}$, so by definition of ϕ and (3.3) it follows that

$$\begin{aligned} d(u, f(u,u,\ldots,u)) &\ll \frac{c}{k+1} + \lambda\phi\left(\frac{c}{k+1}, \frac{c}{k+1}, \ldots, \frac{c}{k+1}\right) \\ &\quad + \lambda\phi\left(\frac{c}{k+1}, \frac{c}{k+1}, \ldots, \frac{c}{k+1}, \frac{c}{k+1}\right) \\ &\quad + \cdots + \lambda\phi\left(\frac{c}{k+1}, \frac{c}{k+1}, \ldots, \frac{c}{k+1}\right) \\ &\preceq \frac{c}{k+1} + \lambda\frac{c}{k+1} + \lambda\frac{c}{k+1} + \cdots + \lambda\frac{c}{k+1} \\ &\preceq c \end{aligned}$$

for all $n > n_1$. Therefore by c) of Remark 2.1, we obtain $d(u, f(u,u,\ldots,u)) = \theta$ i.e. $f(u,u,\ldots,u) = u$. Thus u is required fixed point of f. □

If we take $\phi(x_1, x_2, \ldots, x_k) = \max\{x_1, x_2, \ldots, x_k\}$ in above theorem we get the following extension of result of Ćirić and Prešić [5] in ordered cone metric spaces.

Corollary 3.1. *Let $(X, \sqsubseteq, d)$ be an ordered complete cone metric space. Let $f\colon X^k \to X$ be a nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following conditions hold:*

i) *f is ordered Ćirić-Prešić type contraction;*

ii) *there exist $x_1, x_2, \ldots, x_k \in X$ such that $x_k \sqsubseteq f(x_1, x_2, \ldots, x_k)$, $x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_k$ and $\mu = \max\left\{\dfrac{d(x_1, x_2)}{\delta}, \dfrac{d(x_2, x_3)}{\delta^2}, \ldots, \dfrac{d(x_k, f(x_1, x_2, \ldots, x_k))}{\delta^k}\right\}$ exists in E where $\delta = \lambda^{\frac{1}{k}}$;*

iii) *if a nondecreasing sequence $\{x_n\}$ converges to $x \in X$, then $x_n \sqsubseteq x$ for all $n \in \mathbb{N}$.*

Then f has a fixed point $u \in X$.

The following example shows that a mapping on an ordered cone metric space can be an "ordered" ϕ-Ćirić-Prešić type contraction, while it is not a contraction in Ćirić-Prešić [5] sense.

Example 3.1. Let $X = [0, 2]$ and let order relation $\sqsubseteq$ be defined by

$$\sqsubseteq = \{(x, y) : x, y \in [0, 1), \text{ with } y \leq x\} \cup \{(x, y) : x, y \in [1, 2], \text{ with } y \leq x\}.$$

Let $E = C^1_{\mathbb{R}}[0, 1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E : x(t) \geq 0 \text{ on } [0, 1]\}$ (this cone is not normal). Define $d\colon X \times X \to E$ by $d(x, y) = |x - y|\varphi$ where $\varphi\colon [0, 1] \to \mathbb{R}$ such that $\varphi(t) = e^t$. It is easy to see that d is a cone metric on X. Let $f\colon X^2 \to X$ be a mapping defined by

$$f(x, y) = \frac{x + y}{4} + \frac{1}{2}, \text{ if } (x, y) \in [0, 1) \times [0, 1), \text{ or } (x, y) \in [1, 2] \times [1, 2],$$
$$f(x, y) = x + y - 1, \text{ if } (x, y) \in [0, 1) \times [1, 2], \text{ or } (x, y) \in [1, 2] \times [0, 1),$$

and $\phi(x, y) = \max\{x, y\}$. Then f is nondecreasing with respect to $\sqsubseteq$ and an ordered ϕ-Ćirić-Prešić type contraction i.e.

$$d(f(x, y), f(y, z)) \preceq \lambda \max\{d(x, y), d(y, z)\}, \tag{3.4}$$

for all $x, y, z \in X$ with $x \sqsubseteq y \sqsubseteq z$, and for some $\lambda \in (0, 1)$. Indeed we have to check validity of above inequality only for $x, y, z \in [0, 1)$ and $x, y, z \in [1, 2]$. For example if $x, y, z \in (0, 1]$ with $x \sqsubseteq y \sqsubseteq z$ i.e. $z \leq y \leq x$ then

$$\begin{aligned} d(f(x, y), (y, z)) &= |\frac{x + y}{4} - \frac{y + z}{4}|e^t \\ &= |\frac{x - y}{4} + \frac{y - z}{4}|e^t \\ &\preceq \frac{1}{4}(|x - y| + |y - z|)e^t \\ &\preceq \frac{1}{2}\max\{d(x, y), d(y, z)\}. \end{aligned}$$

Similarly inequality holds good if $x, y, z \in [1, 2]$ with $\lambda \in \left[\dfrac{1}{2}, 1\right)$. Note that all the conditions of Theorem 3.1 are satisfied and $f(1, 1) = 1$. On the other hand, f is not a contraction in

Ćirić-Prešić [5] sense. For example if $x = y = 1, z = \frac{1}{2}$, we have

$$d(f(x,y),f(y,z)) = d\Big(f(1,1),f(1,\frac{1}{2})\Big) = d\Big(\frac{1+1}{4}+\frac{1}{2},1+\frac{1}{2}-1\Big) = d\Big(1,\frac{1}{2}\Big) = \frac{1}{2}e^t,$$

and

$$\max\{d(x,y),d(y,z)\} = \max\Big\{|1-1|e^t,\Big|1-\frac{1}{2}\Big|e^t\Big\} = \max\Big\{0,\frac{1}{2}e^t\Big\} = \frac{1}{2}e^t.$$

Therefore it is not possible to find a constant λ with $\lambda \in (0,1)$ such that

$$d(f(x,y),f(y,z)) \preceq \lambda \max\{d(x,y),d(y,z)\} \text{ for all } x,y,z \in X.$$

Thus we can not apply even the cone metric version of Theorem 1.1 or 1.2 to conclude the existence of fixed point of f.

In above example fixed point is unique, but it need not to be unique as shown in the following example.

Example 3.2. Let $X = \{1,2,3,4\}$ and let order relation be defined by

$$\sqsubseteq = \{(1,1),(2,2),(3,3),(4,4),(1,3),(1,4)\}.$$

Let $E = \mathbb{R}^2$ with usual norm and $P = \{(a,b) : a \geq 0, b \geq 0\}$. Define $d =: X \times X \to E$ by $d(x,y) = |x-y|(1,\alpha)$, where $\alpha \geq 0$. It is easy to see that d is a cone metric on X. Let $f\colon X^2 \to X$ be a mapping defined by

$$f(1,1) = 1, f(2,2) = 2, f(3,3) = 1, f(4,4) = 3,$$
$$f(x,y) = \min\{x,y\} \text{ for all } x,y \in X, \text{ with } x \neq y,$$

and $\phi(x,y) = \max\{x,y\}$. Then f is nondecreasing with respect to $\sqsubseteq$ and an ordered ϕ-Ćirić-Prešić type contraction. Indeed we have to check the validity of inequality (3.4) only for $x = 1, y = 4, z = 4$, and it is equivalent to

$$\begin{aligned} d(f(x,y),f(y,z)) &\preceq \lambda \max\{d(x,y),d(y,z)\} \\ &\Leftrightarrow\ d(f(1,4),f(4,4)) \preceq \lambda \max\{d(1,4),d(4,4)\} \\ &\Leftrightarrow\ 2(1,\alpha) \preceq 3\lambda(1,\alpha), \end{aligned}$$

which is satisfied for $\lambda \in \Big[\frac{2}{3},1\Big)$. All the conditions of Theorem 3.1 hold, and $f(1,1) = 1$, $f(2,2) = 2$. Again f is not a contraction in Ćirić-Prešić [5] sense. For example if $x = 1$, $y = 3$, $z = 4$, we have

$$d(f(x,y),f(y,z)) = d(f(1,3),f(3,4)) = d(1,3) = 2(1,\alpha),$$
$$\max\{d(x,y),d(y,z)\} = \max\{d(1,3),d(3,4)\} = 2(1,\alpha).$$

Therefore it is not possible to find a constant λ with $\lambda \in (0,1)$ such that

$$d(f(x,y),f(y,z)) \preceq \lambda \max\{d(x,y),d(y,z)\} \text{ for all } x,y,z \in X.$$

Following corollary is an extension of result of Prešić [20] in ordered cone metric spaces.

Corollary 3.2. *Let $(X,\sqsubseteq,d)$ be an ordered complete cone metric space. Let $f\colon X^k \to X$ be a nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following conditions hold:*

i) *f is ordered Prešić type contraction;*

ii) *there exist* $x_1, x_2, \ldots, x_k \in X$ *such that* $x_k \sqsubseteq f(x_1, x_2, \ldots, x_k)$, $x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_k$ *and* $\mu = \max\left\{\frac{d(x_1,x_2)}{\delta}, \frac{d(x_2,x_3)}{\delta^2}, \ldots, \frac{d(x_k, f(x_1,x_2,\ldots,x_k))}{\delta^k}\right\}$ *exists in* E *where* $\delta = \lambda^{\frac{1}{k}}$;

iii) *if a nondecreasing sequence* $\{x_n\}$ *converges to* $x \in X$, *then* $x_n \sqsubseteq x$ *for all* $n \in \mathbb{N}$.

Then f *has a fixed point* $u \in X$.

Acknowledgment. The authors would like to thank the anonymous referee for his/her comments that helped us to improve this article.

REFERENCES

[1] M. Abbas and B. E. Rhoades: *Fixed and periodic point results in cone metric spaces*, Appl. Math. Lett., **21**(2008), 511-515.

[2] C. D. Aliprantis and R. Tourky: *Cones and duality*, Graduate Studies in Mathematics 84, American Mathematicsl Society, Providence, Rhode Island, 2007.

[3] I. Altun and B. Damnjanović and D. Djorić: *Fixed point and common fixed point theorems on ordered cone metric spaces*, Appl. Math. Lett., **23**(2010), No. 3, 310-316.

[4] I. Altun and G. Durmaz: *Some fixed point theorems on ordered cone metric spaces*, Rend. Circ. Mat. Palermo, **58**(2009), 319-325.

[5] Lj. B. Ćirić and S. B. Prešić: *On Prešić type generalisation of Banach contraction principle*, Acta. Math. Univ. Com., **LXXVI**(2007), No. 2, 143-147.

[6] K. Deimling: *Nonlinear Functional Analysis*, Springer-Verlag, 1985.

[7] H.-S. Ding, Z. Kadelburg, E. Karapinar and S. Radenović: *Common fixed points of weak contractions in cone metric spaces*, Abstr. Appl. Anal., Volume 2012, Article ID 793862, 18 pages.

[8] R. George, K. P. Reshma and R. Rajagopalan: *A generalised fixed point theorem of Prešić type in cone metric spaces and application to Markov process*, Fixed Point Theory Appl., Volume 2011, Article ID 2011:85.

[9] L. G. Huang and X. Zhang: *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., **332**(2007), No. 2, 1468-1476.

[10] D. Illić. and V. Rakočević: *Common fixed points for maps on cone metric spae*, J. Math. Anal. Appl., **341**(2008), 876-882.

[11] D. Illić. and V. Rakočević: *Quasi-contraction on a cone metric space*, Appl. Math. Lett., **22**(2009), 728-731.

[12] S. Janković, Z. Kadelburg and S. Radenović: *On cone metric spaces: A survey*, Nonlinear Anal., **74**(2011), 2591-2601.

[13] G. Jungck, S. Radenović, S. Radojević and V. Rakočević: *Common fixed point theorems for weakly compatible pairs on cone metric spaces*, Fixed Point Theory Appl., **57**(2009), article ID 643840, 13 pages.

[14] Z. Kadelburg, Mirjana Pavlović and S. Radenović: *Common fixed point theorems for ordered contractions and quasicontractions in ordered cone metric spaces*, Comput. Math. Appl., **59**(2010), 3148-3159.

[15] Z. Kadelburg, S. Radenović and V. Rakočević: *A note on the equivalence of some metric and cone metric fixed fixed point results*, Appl. Math. Lett., **24**(2011), 370-374.

[16] L. V. Kantorovich: *On some further applications of the newton approximation method*, Vestn. Leningr. Univ. Ser. Mat. Mekah. Astron., **12**(1957), No. 7, 68-103.

[17] Đ. R. Kurepa: *Tableaux ramifiés d'ensembles. Espaces pseudo-distanciés*, C. R. Acad. Sci., Paris, **198**(1934), 1563-1565.

[18] Đ. R. Kurepa: *Free power or width of some kinds of mathematical structure*, Publ. Inst. Math., Nouv. Sér., **42**(1987), 3-12.

[19] J. J. Nieto and R. R. Lopez: *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equation*, Order, (2005), 223-239.

[20] S. B. Prešić: *Sur la convergence des suites*, C. R. Acad. Sci., Paris, **260**(1965), 3828-3830.

[21] P. D. Proinov: *A unified theory of cone metric spaces and its applications to the fixed point theory*, ArXiv:111.4920, 2011, 51 pages.

[22] A. C. M. Ran and M. C. B Reurings: *A fixed point theorem in partially ordered sets and some application to matrix equations*, Proc. Amer. Math. Soc., **132**(2004), 1435-1443.

[23] Sh. Rezapour and R. Hamlbarani: *Some notes on the paper Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., **345**(2008), No. 2, 719-724.

[24] J. S. Vandergraft: *Newton's method for convex operators in partially ordered spaces*, SIAM J. Numer. Anal., **4**(1967), 406-432.

[25] P. Vetro: *Common fixed points in cone metric spaces*, Rend. Circ. Mat. Palermo, **56**(2007), No. 2, 464-468.

[26] P. P. Zabreĭko: *K-metric and K-normed spaces: survey*, Collect. Math., **48**(1997), No. 4-6, 825-859.

Govt. S.G.S.P.G. College
Department of Mathematics
Ganj Basoda, Distt. Vidisha (M.P.), India.

Shri Vaishnav Institute of Technology & Science
Department of Applied Mathematics
Gram Baroli, Sanwer Road, Indore (M.P.), 453331 *India.*
E-mail address: satishmathematics@yahoo.co.in

Shri Vaishnav Institute of Technology & Science
Department of Applied Mathematics
Gram Baroli, Sanwer Road, Indore (M.P.), 453331 *India.*