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COINCIDENCE AND FIXED POINTS OF WEAKLY CONTRACTIVE MAPS

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Abstract. Weakly contractive maps introduced by Alber and Guerre-Delabriere [4], is a wider class of maps which contains the classical Banach contraction as a special case and is closely related to the nonlinear contractions of Boyd-Wong and Reich type. The first section of this paper is introductory in nature and captures the motivation for the current work. In section 2, we study the existence of coincidence and fixed points of Ćirić type weakly generalized contractions in metric spaces and some recent results are discussed as special cases. Finally, in section 3, applications regarding the convergence theorems for modified Mann iterations and modified Ishikawa iterations in a convex metric space are also discussed.

1. INTRODUCTION

In all that follows (X, d) is a metric space. A self-map T of X is a Banach contraction if there exists $k \in [0, 1)$ such that for each $x, y \in X$,

$$d(Tx, Ty) \leq kd(x, y). \tag{1.1}$$

The classical Banach contraction principle (BCP) states that a self-map T of a complete metric space admits a unique fixed point if T is a Banach contraction. Since its inception, the BCP has been cardinal in the field of nonlinear analysis and finds numerous applications in Mathematical Sciences and Engineering (see, for instance, Agarwal et al. [2], Agarwal et al. [3], Kirk and Sims [26] and Zeidler [39]). For some fundamental generalizations of the condition (1.1) and their comprehensive comparison, one may refer to Rhoades [32] (see also [33]).

Alber and Guerre-Delabriere [4] generalized the condition (1.1) and coined the concept of weakly contractive maps.

Definition 1.1. [34] A self-map T of X is *weakly contractive* if for each $x, y \in X$,

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)), \tag{1.2}$$

where $\phi\colon [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing such that ϕ is positive on $(0, \infty)$ and $\phi(0) = 0$.

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Alber and Guerre-Delabriere [4] used the concept of weakly contractive maps for obtaining their results in the setting of Hilbert spaces. However, they remarked that their results are true for uniformly smooth and uniformly convex Banach spaces (see [34], p. 2684). Rhoades [34] extended some of the work of Alber and Guerre-Delabriere [4] to arbitrary Banach spaces and obtained an interesting generalization of the BCP in the framework of a metric space for weakly contractive maps. Precisely, he obtained the following result.

Theorem 1.1. *Let T be a weakly contractive self-map of X then T has a unique fixed point provided that X is complete.*

Notice that weakly contractive maps contain contractions, as the special case ($\phi(t) = (1-k)t$). Taking another self-map into consideration and using the idea of Goebel [19] and Jungck [22], the concept of weakly contractive maps was extended to two self-maps independently by Beg and Abbas [8] and Song [36].

Definition 1.2. [36] A self-map T of X is *f-weakly contractive* if for each $x, y \in X$,

$$d(Tx, Ty) \leq d(fx, fy) - \phi(d(fx, fy)), \tag{1.3}$$

where f is a self-map of X and $\phi\colon [0,\infty) \to [0,\infty)$ is lower semi-continuous from the right such that ϕ is positive on $(0,\infty)$ and $\phi(0) = 0$.

We remark that Definition 1.2 with ϕ continuous is due to Beg and Abbas [8, Def 1.1]. Further, if f is the identity map on X, and ϕ is a continuous and nondecreasing function in Definition 1.2 then a f-weakly contractive map reduces to a weakly contractive map studied by Alber and Guerre-Delabriere [4] and Rhoades [34], and if $\phi(t) = (1-k)t$, for a constant k with $0 < k < 1$, then a f-weakly contractive map is an f-contraction (see [6] and [36]). Notice that if $f = I$ (the identity map on X) and ϕ is a lower semi-continuous map from the right then $\varphi(t) = t - \phi(t)$ is upper semi-continuous from the right, then (1.3) becomes

$$d(Tx, Ty) \leq \varphi(d(x, y)). \tag{1.4}$$

Therefore, weakly contractive maps for which ϕ is lower semi-continuous from the right are of Boyd and Wong [12] type (see also [6] and [36]). Further, notice that with $f = I$, if we define $k(t) = 1 - \dfrac{\phi(t)}{t}$ for $t > 0$ and $k(0) = 0$ then the condition (1.3) is reduced to the following Reich type [31] contractive condition:

$$d(Tx, Ty) \leq k(d(fx, fy))d(fx, fy). \tag{1.5}$$

Indeed, for a suitable choice of $k(t)$, for $t > 0$ and $k(0) = 0$, weakly contractive maps reduce to (1.5). For details, refer to [6] and [36].

Abbas and Doric [1], Azam and Shakeel [5], Bae [6], Beg and Abbas [8], Bose and Roychowdhury [11], Ćirić, Hussian and Cakic [15], Doric [17], Dutta and Choudhury [18], Popescu [29], Song [36] and Zhang and Song [40], among others, obtained coincidence and fixed point theorems for weakly contractive maps.

The following result is due to Beg and Abbas [8].

Theorem 1.2. *Let T and f be self-maps of X such that $TX \subseteq fX$ and* (1.3) *holds for all $x, y \in X$, where $\phi\colon [0,\infty) \to [0,\infty)$ is continuous and nondecreasing such that ϕ is positive on $(0,\infty)$ and $\phi(0) = 0$. If $f(X)$ is a complete subspace of X then T and f have a common fixed point provided that T and f are commuting at their coincidence points.*

Azam and Shakeel [5] obtained a similar result for three self-maps of X satisfying the following condition for each $x, y \in X$,

$$d(Sx, Ty) \leq d(fx, fy) - \phi(d(fx, fy)), \tag{1.6}$$

where $\phi\colon [0,\infty) \to [0,\infty)$ is continuous and nondecreasing such that ϕ is positive on $(0,\infty)$ and $\phi(0) = 0$.

The following generalized contraction for a self-map T of X is essentially due to Ćirić [14] which is listed as $(21')$ by Rhoades [32]:

$$d(Tx, Ty) \leq km(x, y), \tag{1.7}$$

where $0 < k < 1$ and

$$m(x, y) = \max\Big\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\Big\}.$$

Consider the following Ćirić type weakly generalized contraction for a self-map T of X

$$d(Tx, Ty) \leq m(x, y) - \phi(m(x, y)), \tag{1.8}$$

where $\phi : [0,\infty) \to [0,\infty)$ is lower semi-continuous from the right such that ϕ is positive on $(0,\infty)$ and $\phi(0) = 0$.

Zhang and Song [40, Th. 2.1], have shown the existence of a unique common fixed point of self-maps S and T of a complete metric space X satisfying the following condition for each $x, y \in X$:

$$d(Sx, Ty) \leq m'(x, y) - \phi(m'(x, y)), \tag{1.9}$$

where ϕ is as in (1.8), and

$$m'(x, y) = \max\Big\{d(x, y), d(x, Sx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Sx)]\Big\}.$$

See also the remark following Theorem 2.3 below. Notice that (1.9) with $S = T$ is the condition (1.8).

Using an additional function $\psi\colon [0,\infty) \to [0,\infty)$ and a quadruplet of maps Abbas and Doric [1] obtained the following common fixed point theorem in a complete metric space.

Theorem 1.3. *Let S, T, f and g be self-maps of a complete metric space (X, d) such that*
(i) $TX \subseteq fX, SX \subseteq gX$,
(ii) *one of the ranges SX, TX, fX and gX is closed,and if for each $x, y \in X$,*
(iii) $\psi(d(Sx, Ty)) \leq \psi(M(x, y)) - \phi(M(x, y))$, *where*

$$M(x, y) := \max\Big\{d(fx, gy), d(fx, Sx), d(gy, Ty), \frac{1}{2}[d(fx, Ty) + d(gy, Sx)]\Big\},$$

ϕ is as in (1.8) *and $\psi\colon [0,\infty) \to [0,\infty)$ is continuous and nondecreasing with $\psi(t) = 0$ if and only if $t = 0$.*

Then S, T, f and g have a unique common fixed point in X provided that the pairs (S, f) and (T, g) are weakly compatible.

We remark that completeness of the space X in Theorem 1.3 implies that closed ranges considered in (ii) are also complete.

In Section 2 of this paper, first we obtain two variants of Theorem 1.3. Indeed, we obtain coincidence and common fixed point theorems for a pair, triplet and quadruplet of Ćirić type weakly generalized contractions on a metric space, wherein the completeness of the metric space X is replaced by much weaker alternative hypotheses. This is shown,

as another alternative, that if the space X is complete in Theorem 1.3, the requirement (ii) is not needed provided that one of the maps S, T, f and g is continuous (cf. Theorem 2.2). Usefulness of the results is illustrated by examples to clarify underlying strength and distinctions. In section 3, applications regarding the convergence of modified Mann and modified Ishikawa iterations for weakly contractive maps in convex spaces are discussed.

2. COINCIDENCE AND FIXED POINT THEOREMS

For the sake of brevity, we follow the following notations, wherein S, T, f and g are maps to be defined specifically in a particular context, while x and y are the elements of the domain X. Throughout this paper, let $C(S, f) = \{v : Sv = fv\}$ denote the collection of coincidence points of self-maps S and f of a metric space X:

$$M(x,y) := \max\Big\{d(fx,gy), d(fx,Sx), d(gy,Ty), \frac{1}{2}[d(fx,Ty) + d(gy,Sx)]\Big\},$$

$$M(f,T) := \max\Big\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{1}{2}[d(fx,Ty) + d(fy,Tx)]\Big\},$$

$$m(f,T) := \max\{d(fx,fy), d(fx,Tx), d(fy,Ty)\},$$

$\Phi = \{\phi | \phi\colon [0,\infty) \to [0,\infty)$ is lower semi-continuous from the right such that ϕ is positive on $(0,\infty)$ and $\phi(0) = 0\}$ and

$\Psi = \{\psi | \psi\colon [0,\infty) \to [0,\infty)$ is continuous and nondecreasing with $\psi(t) = 0 \Leftrightarrow t = 0\}$.

The following is our first result for a quadruplet of maps on a metric space.

Theorem 2.1. *Let S, T, f and g be self-maps of a metric space X such that*

$$TX \subseteq fX \text{ and } SX \subseteq gX, \tag{2.1}$$

if for each $x, y \in X$,

$$\psi(d(Sx,Ty)) \le \psi(M(x,y)) - \phi(M(x,y)), \tag{2.2}$$

where $\phi \in \Phi$ and $\psi \in \Psi$. If one of fX, gX, SX or TX is a complete subspace of X then:

(I) *$C(S,f)$ and $C(T,g)$ are nonempty. Further,*

(II) *S and f have a common fixed point provided that they commute just at a coincidence point;*

(III) *T and g have a common fixed point provided that they commute just at a coincidence point;*

(IV) *S, T, f and g have a unique common fixed point provided* (II) *and* (III) *are true.*

Proof. Pick $x_0 \in X$. As $TX \subseteq fX$, we can choose a point $x_1 \in X$ such that $Tx_0 = fx_1$, and for this point x_1, there exists a point x_2 in X such that $Sx_1 = gx_2$. Continuing in this manner, we define two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{2n} = Tx_{2n} = fx_{2n+1}$ and $y_{2n+1} = Sx_{2n+1} = gx_{2n+2}$, $n = 0, 1, 2, \ldots$.

In view of Abbas and Doric [1, Th. 2.1], the sequence $\{y_n\}$ is Cauchy. Assume that fX is complete. Notice that the sequence $\{y_{2n}\}$ is contained in fX and has a limit in fX. Call it u. Let $v \in f^{-1}u$. Then $fv = u$. The subsequence $\{y_{2n+1}\}$ also converges to u. Now

we show that $Sv = fv = u$. By (2.2),

$$\psi(d(Sv, y_{2n})) = \psi(d(Sv, Tx_{2n})) \leq \psi\Big(\max\Big\{d(fv, gx_{2n}), d(fv, Sv), d(gx_{2n}, Tx_{2n}), \frac{1}{2}[d(fv, Tx_{2n}) + d(gx_{2n}, Sv)]\Big\}\Big) - \phi\Big(\max\Big\{d(fv, gx_{2n}), d(fv, Sv), d(gx_{2n}, Tx_{2n}), \frac{1}{2}[d(fv, Tx_{2n}) + d(gx_{2n}, Sv)]\Big\}\Big).$$

Making $n \to \infty$,

$$\psi(d(Sv, u)) \leq \psi\Big(\max\Big\{d(fv, u), d(fv, Sv), d(u, u), \frac{1}{2}[d(fv, u) + d(u, Sv)]\Big\}\Big) - \phi\Big(\max\Big\{d(fv, u), d(fv, Sv), d(u, u), \frac{1}{2}[d(fv, u) + d(u, Sv)]\Big\}\Big),$$

that is $\psi(d(Sv, fv)) = \psi(d(Sv, u)) \leq \psi(d(fv, Sv)) - \phi(d(fv, Sv))$, a contradiction.

Therefore, $Sv = fv = u$. This proves that $C(S, f)$ is nonempty.

Since $SX \subseteq gX$, $Sv = u$ implies $u \in gX$. Let $w \in g^{-1}u$. Then $gw = u$. Now we show that $Tw = u$. If $Tw \neq u$,

$$\psi(d(u, Tw)) = \psi(d(Sv, Tw)) \leq \psi\Big(\max\Big\{d(fv, gw), d(fv, Sv), d(gw, Tw), \frac{1}{2}[d(fv, Tw) + d(gw, Sv)]\Big\}\Big) - \phi\Big(\max\Big\{d(fv, gw), d(fv, Sv), d(gw, Tw), \frac{1}{2}[d(fv, Tw) + d(gw, Sv)]\Big\}\Big).$$

Therefore, $\psi(d(Sv, Tw)) \leq \psi(d(Sv, Tw)) - \phi(d(Sv, Tw))$, a contradiction.

Hence $Tw = u$. So $Tw = gw = u = fv = Sv$. This proves that $C(T, g)$ is nonempty. If gX is complete then an analogous argument establishes (I). If SX is complete then $u \in SX \subset gX$ and we see that $C(T, g)$ is nonempty. Similarly if TX is complete then $u \in TX \subset fX$, and $C(S, f)$ is nonempty. Thus (I) is completely established.

Now suppose that S and f are commuting at their coincidence point v. Then $SSv = Sfv = fSv = ffv$. So, by (2.2), we have

$$\begin{aligned}\psi(d(Sv, SSv)) = \psi(d(SSv, Tw)) &\leq \psi\Big(\max\Big\{d(fSv, gw), d(fSv, SSv), d(gw, Tw), \frac{1}{2}[d(fSv, Tw) + d(gw, SSv)]\Big\}\Big) - \phi\Big(\max\Big\{d(fSv, gw), d(fSv, SSv), d(gw, Tw), \frac{1}{2}[d(fSv, Tw) + d(gw, SSv)]\Big\}\Big)\\ &= \psi(d(SSv, Tw)) - \phi(d(SSv, Tw)),\end{aligned}$$

a contradiction. This proves (II). Analogously, if T and g are commuting at their coincidence point w, then w is a common fixed point of T and g. This proves (III). Now, (IV) is immediate.

In order to see the uniqueness part of (IV), assume that there are two distinct common fixed points y and z, that is $fy = gy = Sy = Ty = y$ and $fz = gz = Sz = Tz = z$. Using

(2.2),

$$\begin{aligned}\psi(d(y,z)) = \psi(d(Sy,Tz)) &\le \psi\Big(\max\Big\{d(fy,gz), d(fy,Sy), d(gz,Tz),\\ &\quad \frac{1}{2}[d(fy,Tz)+d(gz,Sy)]\Big\}\Big) - \phi\Big(\max\Big\{d(fy,gz),\\ &\quad d(fy,Sy), d(gz,Tz), \frac{1}{2}[d(fy,Tz)+d(gz,Sy)]\Big\}\Big)\\ &\le \psi(d(y,z)) - \phi(d(y,z)) < \psi(d(y,z)),\end{aligned}$$

a contradiction. This completes the proof. □

Self-maps S and f of a metric space X are compatible if $\lim_{n\to\infty} d(Sfx_n, fSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} fx_n = t$ for some $t \in X$, see Jungck [23]. For a detailed comparison of various weaker forms of commuting maps, one refer to Singh and Tomar [35].

The following result is another variant of Theorem 1.3.

Theorem 2.2. *Let S, T, f and g be self-maps of a complete metric space (X,d) such that conditions* (2.1) *and* (2.2) *of Theorem* 2.1 *are satisfied. If the pairs (S,f) and (T,g) are compatible and one of the maps S, T, f and g is continuous then S, T, f and g have a unique common fixed point in X.*

Proof. As in Theorem 2.1, the sequence $\{y_n\}$ defined by

$$y_{2n} = Tx_{2n} = fx_{2n+1} \text{ and } y_{2n+1} = Sx_{2n+1} = gx_{2n+2}, \quad n = 0,1,2,\ldots$$

is a Cauchy sequence. By completeness of X, sequence $\{y_n\}$ converges to a point $u \in X$. Also

$$y_{2n} = Tx_{2n} = fx_{2n+1} \to u, y_{2n+1} = Sx_{2n+1} = gx_{2n+2} \to u.$$

Suppose that f is continuous. Then $ffx_{2n} \to fu$, $fSx_{2n} \to fu$, and the compatibility of (S,f) implies $Sfx_{2n} \to fu$. Using (2.2), we have

$$\psi(d(Sfx_{2n}, Tx_{2n})) \le \psi(M(fx_{2n}, x_{2n})) - \phi(M(fx_{2n}, x_{2n})).$$

Making $n \to \infty$,

$$\begin{aligned}\psi(d(fu,u)) &\le \psi\Big(\max\Big\{d(fu,u), d(fu,fu), d(u,u), \frac{1}{2}[d(fu,u)+d(u,fu)]\Big\}\Big)\\ &\quad -\phi\Big(\max\Big\{d(fu,u), d(fu,fu), d(u,u), \frac{1}{2}[d(fu,u)+d(u,fu)]\Big\}\Big),\end{aligned}$$

that is $\psi(d(fu,u)) \le \psi(d(fu,u)) - \phi(d(fu,u))$, a contradiction. So $fu = u$.

Further, from (2.2), $\psi(d(Su,Tx_{2n})) \le \psi(M(u,x_{2n})) - \phi(M(u,x_{2n}))$. Making $n \to \infty$,

$$\begin{aligned}\psi(d(Su,u)) &\le \psi\Big(\max\Big\{d(fu,u), d(fu,Su), d(u,u), \frac{1}{2}[d(fu,u)+d(u,Su)]\Big\}\Big)\\ &\quad -\phi\Big(\max\Big\{d(fu,u), d(fu,Su), d(u,u), \frac{1}{2}[d(fu,u)+d(u,Su)]\Big\}\Big)\\ &= \psi(d(Su,u)) - \phi(d(Su,u),\end{aligned}$$

a contradiction. Therefore, $Su = u$.

Since $SX \subseteq gX$, there exists a point $v \in X$ such that $u = fu = Su = gv$. We claim that $u = Tv$. If not, then using (2.2),

$$\begin{aligned}\psi(d(Su,Tv)) &\leq \psi\Big(\max\Big\{d(fu,gv), d(fu,Su), d(gv,Tv), \frac{1}{2}[d(fu,Tv)+d(gv,Su)]\Big\}\Big)\\ &\quad -\phi\Big(\max\Big\{d(fu,gv), d(fu,Su), d(gv,Tv), \frac{1}{2}[d(fu,Tv)+d(gv,Su)]\Big\}\Big)\\ &= \psi(d(u,Tv)) - \phi(d(u,Tv)),\end{aligned}$$

a contradiction. Hence $u = Tv = gv$.

Now compatibility of (T, g) implies that $Tu = Tgv = gTv = gu$. From (2.2), we get

$$\begin{aligned}\psi(d(Su,Tu)) &\leq \psi\Big(\max\Big\{d(fu,gu), d(fu,Su), d(gu,Tu), \frac{1}{2}[d(fu,Tu)+d(gu,Su)]\Big\}\Big)\\ &\quad -\phi\Big(\max\Big\{d(fu,gu), d(fu,Su), d(gu,Tu), \frac{1}{2}[d(fu,Tu)+d(gu,Su)]\Big\}\Big),\end{aligned}$$

that is $\psi(d(u,Tu)) \leq \psi(d(u,Tu)) - \phi(d(u,Tu))$, a contradiction. So $u = Tu = Su = fu = gu$. This proves that S, T, f and g have a common fixed point. This result holds if g is continuous instead of f. Suppose S is continuous then the compatibility of (S, f) gives $Sf_{2n} = SSx_{2n} = fSx_{2n} \to Su$.

Using (2.2), we get $\psi(d(SSx_{2n}, Tx_{2n})) \leq \psi(M(Sx_{2n}, x_{2n})) - \phi(M(Sx_{2n}, x_{2n}))$. Making $n \to \infty$,

$$\begin{aligned}\psi(d(Su,u)) &\leq \psi\Big(\max\Big\{d(Su,u), d(Su,Su), d(u,u), \frac{1}{2}[d(Su,u)+d(u,Su)]\Big\}\Big)\\ &\quad -\phi\Big(\max\Big\{d(Su,u), d(Su,Su), d(u,u), \frac{1}{2}[d(Su,u)+d(u,Su)]\Big\}\Big)\\ &= \psi(d(Su,u)) - \phi(d(Su,u)),\end{aligned}$$

a contradiction. Hence $u = Su$.

Since $SX \subseteq gX$, there exists a point v in X such that $u = Su = gv$. We now show that $Tv = gv$. By (2.2), $\psi(d(SSx_{2n}, Tv)) \leq \psi(M(Sx_{2n}, v)) - \phi(M(Sx_{2n}, v))$. Letting $n \to \infty$,

$$\begin{aligned}\psi(d(Su,Tv)) &\leq \psi\Big(\max\Big\{d(Su,gv), d(Su,Su), d(Su,Tv), \frac{1}{2}[d(Su,Tv)+d(Su,Su)]\Big\}\Big)\\ &\quad -\phi\Big(\max\Big\{d(Su,gv), d(Su,Su), d(Su,Tv), \frac{1}{2}[d(Su,Tv)+d(Su,Su)]\Big\}\Big),\end{aligned}$$

that is $\psi(d(u,Tv)) = \psi(d(u,Tv)) - \phi(d(u,Tv))$, a contradiction. Therefore, $u = Tv = gv = Su$, and the compatibility of (T, g) yields $Tu = Tgv = gTv = gu$.

Now using (2.2), we get $\psi(d(Sx_{2n}, Tu)) \leq \psi(M(x_{2n}, u)) - \phi(M(x_{2n}, u))$. For $n \to \infty$, $\psi(d(u,Tu)) = \psi(d(u,Tu)) - \phi(d(u,Tu))$. So $u = Tu = gu$.

As $TX \subseteq fX$, there exists a point w in X such that $u = Tu = gu = fw$.

We now show that $u = Sw$.

Using (2.2) with $x = w$ and $y = u$, we have $\psi(d(Sw,Tu)) = \psi(M(w,u)) - \phi(M(w,u)) = \psi(d(u,Sw)) - \phi(d(u,Sw))$, a contradiction. Therefore, $u = Sw = fw = Tu = gu$.

Compatibility of the pair (S, f) implies $u = Sw = Sfw = fSw = fu$. Hence $u = Su = Tu = gu = fu$. Thus u is a common fixed point of S, T, f and g. If T is continuous then analogous argument establishes the existence of common fixed point of S, T, f and g. Uniqueness of the fixed point follows easily. □

In case $f = g$ in Theorem 2.1, we obtain a slightly improved version which we state below.

Theorem 2.3. *Let S, T and f be self-maps of a metric space X such that $SX \cup TX \subseteq fX$, and the condition (2.2) with $f = g$ holds. If one of SX, TX or fX is a complete subspace of X. Then S, T and f have a common coincidence point. Further, if f commutes with each of S and T at one of their coincidence points, then S, T and f have a unique common fixed point.*

We remark that Theorem 2.3 with f as the identity map on X is a common fixed point theorem for two self maps S and T of a complete metric space due to Doric [17, Th. 2.1]. Further, Theorem 2.3 with f as the identity map on X and $\psi(t) = t$ for all $t \in [0, \infty)$ is a common fixed point theorem for two self-maps S and T of a complete metric space due to Zhang and Song [40, Th. 2.1]. The following example demonstrates the generality of Theorem 2.3 over results of Azam and Shakeel [5, Theorems 2.3, 2.5], Doric [17] and Zhang and Song [40].

Example 2.1. Let $X = \{1, 2, 3, 4, 5\}$ be endowed with the usual metric. Let $f = g$ and $S1 = S3 = S4 = 1$, $S2 = S5 = 2$, $T1 = T3 = 1$, $T2 = T4 = T5 = 2$ and $f1 = 3$, $f2 = 2$, $f3 = 5$, $f4 = 4$, $f5 = 1$. Notice that SX and TX are contained in fX, and the condition (2.2) of Theorem 2.3 is satisfied, when $\psi\colon [0, \infty) \to [0, \infty)$ and $\phi\colon [0, \infty) \to [0, \infty)$ are respectively defined as $\psi(t) = 2t$ and $\phi(t) = \dfrac{1}{5}t$. So all the hypothesis of Theorem 2.1 (or Theorem 2.1 with $f = g$) are true. Evidently, S, T and f have a unique common fixed point at $x = 2$. Notice that the condition (1.6) used by Azam and Shakeel [5] is not satisfied for $x = 2$ and $y = 1$. Further, it can be easily verified that the condition (1.9) of Zhang and Song [40, Th. 2.1] and condition used by Doric [17, Th. 2.1] are not satisfied for $x = 2$ and $y = 1$ with f as identity map.

We remark that under the hypotheses of Theorem 2.1, the pairs (S, f) and (T, g) may have different coincidence points. Further, if $f = g$ in Theorem 2.1, then maps S, T and f have a common coincidence point (cf. Theorem 2.3). However, it is interesting to note that if f and g are distinct and $S = T$ in Theorem 2.1, then maps f, g and S need not have a common coincidence point. The following examples authenticate these remarks.

Example 2.2. Let $X = [0, \infty)$ be endowed with the usual metric and define

$$\psi, \phi\colon [0, \infty) \to [0, \infty), \quad \psi(t) = t, \quad \phi(t) = \frac{1}{6}t.$$

Let $Sx = x^2 + \dfrac{5}{9}$, $Tx = x^3 + \dfrac{5}{9}$, $fx = 6x^2$, $gx = 6x^3$, $x \in X$. Then:

$$\begin{aligned}\psi(d(Sx, Ty)) &= \psi(|x^2 - y^3|) = |x^2 - y^3| < \psi(6|x^2 - y^3|) - \phi(6|x^2 - y^3|)\\ &= \psi(d(fx, gy) - \phi(d(fx, gy)),\end{aligned}$$

for all $x, y \in X$. So, (2.2) and other hypotheses of Theorem 2.1 are satisfied. We see that $S\left(\dfrac{1}{3}\right) = f\left(\dfrac{1}{3}\right) = \dfrac{6}{9}$ and $T\left(\dfrac{1}{9}\right)^{\frac{1}{3}} = g\left(\dfrac{1}{9}\right)^{\frac{1}{3}} = \dfrac{6}{9}$, that is, S and f have a coincidence at $x = \dfrac{1}{3}$, while T and g have a (different) coincidence at $x = \left(\dfrac{1}{9}\right)^{\frac{1}{3}}$.

The following example shows that if $f = g$ in Theorem 2.1, then maps S, T and f have a common coincidence point as guaranteed by Theorem 2.3. The example also shows that the requirement of commutativity in Theorem 2.3 is imperative and cannot be relaxed.

Example 2.3. Let $X = \{1, 2, 3, 4\}$ be endowed with the usual metric. Let $f = g$ and $S1 = S2 = S3 = S4 = 1$, $T1 = T2 = T3 = 1$, $T4 = 2$ and $f1 = 3$, $f2 = 2$, $f3 = 1$, $f4 = 4$. Then S, T and f satisfy all the hypotheses of Theorem 2.3 with $\psi(t) = 2t$ and $\phi(t) = \frac{1}{3}t$. Evidently, S, T and f have a common coincidence at $x = 3$, while f does not commute with either S or T. So the only common coincidence point 3 of S, T and f is not their common fixed point.

The following example shows that if $S = T$ in Theorem 2.1, then maps f, g and S need not have a common coincidence point.

Example 2.4. Let $X = \{1, 2, 3\}$ be endowed with the usual metric and $S1 = S2 = S3 = 1$, $f1 = 2$, $f2 = 3$, $f3 = 1$ and $g1 = 3$, $g2 = 1$, $g3 = 2$. It is easy to see that S, f and g satisfy all the hypotheses of Theorem 2.1 with $S = T$, $\psi(t) = t$ and $\phi(t) = \frac{1}{2}t$. Notice that $S3 = f3 = 1 = S2 = g2$, that is, S and f have a coincidence at $x = 3$ and S and g have a (different) coincidence at $x = 2$.

From Theorem 2.1 we derive the following result which generalizes fixed point theorems of Chang [13, Th. 8], Jachymiski [21, Th.3.3, Cor. 3.5], Jungck and Pathak [24, Cor. 3.2], Kang and Rhoades [25, Th. 2.3], Song [36, Cor. 3.2], Tan [38, Th. 2] and others.

Corollary 2.1. *Let S, T, f and g be self-maps of a metric space X such that the condition* (2.1) *of Theorem* 2.1 *and the following is satisfied:*

$$d(Sx, Ty) \leq \varphi(M(x, y)), \tag{2.3}$$

for all $x, y \in X$, where $\varphi\colon [0, \infty) \to [0, \infty)$ is upper semi-continuous from the right such that φ is positive on $(0, \infty)$ and $\varphi(0) = 0$. If one of fX, gX, SX or TX is a complete subspace of X then the conclusions of Theorem 2.1 *hold.*

Proof. Let $\phi(t) = t - \varphi(t)$. Then from (2.3), we have $d(Sx, Ty) \leq M(x, y) - \phi(M(x, y))$, for all $x, y \in X$. Then $\phi\colon [0, \infty) \to [0, \infty)$ is lower semi-continuous from the right such that $\phi(t) > 0$ for $t > 0$ and $\phi(0) = 0$. The result follows from the proof of Theorem 2.1 with $\psi(t) = t$. □

For a pair of self-maps T and f of X, we obtain the following result which generalizes certain results of Beg and Abbas [8, Theorem 2.1, 2.5], Dutta and Choudhury [18, Th. 2.1] and Song [36, Theorem 3.1].

Corollary 2.2. *Let T and f be self-maps of a metric space X such that*

$$TX \subseteq fX, \tag{2.4}$$

if for each $x, y \in X$,

$$\psi(d(Tx, Ty)) \leq \psi(M(f, T)) - \phi(M(f, T)), \tag{2.5}$$

where $\phi \in \Phi$ and $\psi \in \Psi$. If one of TX or fX is a complete subspace of X then $C(T, f)$ is nonempty. Further, T and f have a unique common fixed point provided T and f commute at a coincidence point.

Proof. It comes from Theorem 2.3 when $S = T$. Indeed, if $v \in C(T, f)$, then $Tv = fv = u$ (say). If T and f commute at v then u is their unique common fixed point. □

We remark that if f is the identity map on X and $\psi(t) = t$ in Corollary 2.2 then we obtain [40, Cor. 2.2] as a special case. From Corollary 2.2 we immediately obtain the following result.

Corollary 2.3. *Let T and f be self-maps of a metric space X such that the condition* (2.4) *of Corollary* 2.2 *and the following are satisfied:*

$$\psi(d(Tx, Ty)) \leq \psi(m(f, T)) - \phi(m(f, T)) \tag{2.6}$$

for all $x, y \in X$, where $\phi \in \Phi$ and $\psi \in \Psi$. If one of TX or fX is a complete subspace of X. Then conclusions of Corollary 2.2 *are true.*

The following example shows that maps T and f satisfy all the conditions of Corollary 2.2 but the condition (1.3) due to Beg and Abbas [8] and Song [36] is not satisfied.

Example 2.5. Let $X = \{1, 2, 3, 4, 5\}$ with the usual metric and $T1 = T3 = T4 = 1$, $T2 = T5 = 2$ and $f1 = 3$, $f2 = 2$, $f3 = 5$, $f4 = 4$, $f5 = 1$. Notice that the condition (2.5) of Corollary 2.2 is satisfied if $\psi, \phi\colon [0, \infty) \to [0, \infty)$ are defined as $\psi(t) = t$ and $\phi(t) = \frac{1}{5}t$. It can be verified that T and f satisfy all the conditions of Corollary 2.2. Further, it is easy to see that condition (1.3) of Theorem 1.2 is not satisfied for $x = 2$ and $y = 1$.

3. CONVERGENCE THEOREMS

Approximation of fixed points of maps by certain iterative procedures is an interesting area of study in nonlinear analysis. In 1953, Mann [27] introduced the Mann iterative procedure (cf. Definition 3.3 below). Subsequently the Mann iterative procedure was studied on more general settings in many papers. Ishikawa iterative procedure [20] is a highly useful extension of the Mann iterative procedure. For a good survey of these iterations, comparisons and applications, one may refer to Berinde [10]. In this section we obtain certain convergence theorems for modified Mann iterations and modified Ishikawa iterations in a convex metric space. The notion of convexity in metric spaces was first introduced by Takahashi [37] in 1970. He obtained some interesting generalizations of fixed point theorems in Banach spaces (see, for instance [7], [9], [16], [28] and [30]). The following definition is essentially due to Takahashi [37] (see also Agarwal et al. [3], Mishra [28] and Rafiq and Zafar [30]).

Definition 3.1. Let X be a metric space. A mapping $W\colon X \times X \times [0, 1] \to X$ is a *convex structure* on X, if for all $x, y \in X$ and $\lambda \in [0, 1]$, the condition

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$$

is satisfied for all $u \in X$.

A metric space with a convex structure is called a convex metric space. A nonempty subset C of a convex metric space X is convex if $W(x, y, \lambda) \in C$ for all $x, y \in C$ and $\lambda \in [0, 1]$. Notice that all normed spaces and each of their convex subsets are convex metric spaces. However, there are examples of convex metric spaces which are not embedded in a normed space. For details, one may refer to Takahashi [37] and Rafiq and Zafar [30].

Definition 3.2. Consider X be a convex metric space and T, f self-maps of X such that $TX \subseteq fX$. Let $f(X)$ be a convex subset of X. Define a sequence $\{y_n\}$ in $f(X)$ as follows:
$$y_n = fx_{n+1} = W(Tx_n, fx_n, \alpha_n) = (1-\alpha_n)fx_n + \alpha_n Tx_n, x_0 \in X, n \geq 0,$$
where $0 \leq \alpha_n \leq 1$ for each $n \geq 0$. The sequence $\{y_n\}$ so obtained is called *modified Mann iterative scheme.*

Definition 3.3. If we take f the identity map on X in Definition 3.2, then the sequence $\{x_n\}$ is called *Mann sequence* of iterates.

Definition 3.4. Given X a convex metric space, consider T, f self-maps of X such that $TX \subseteq fX$. Let $f(X)$ be a convex subset of X. Define two sequences $\{z_n\}$ and $\{y_n\}$ in $f(X)$ as follows:
$$z_n = fx_{n+1} = W(Tp_n, fx_n, \alpha_n) = (1-\alpha_n)fx_n + \alpha_n Tp_n,$$
$$y_n = fp_n = W(Tx_n, fx_n, \beta_n) = (1-\beta_n)fx_n + \beta_n Tx_n, x_0 \in X, n \geq 0,$$
where $0 \leq \alpha_n, \beta_n \leq 1$ for each $n \geq 0$. The sequence $\{z_n\}$ so obtained is called *modified Ishikawa* iterative scheme.

Theorem 3.1. *Let T and f be self-maps of a convex metric space X such that conditions of Corollary* 2.3 *are satisfied. If TX or fX is a complete convex subspace of X, then the modified Mann iterative scheme converges to a common fixed point of T and f provided that T and f commute at their coincidences.*

Proof. From Corollary 2.3 (see also the proof of Corollary 2.2), we suppose that $v \in C(T, f)$ and $Tv = fv = u$. Suppose T and f commute at their coincidence point v. Then u is their unique common fixed point. Now, we have:
$$\begin{aligned}
\psi(d(y_n, u)) &= \psi(d(fx_{n+1}, fv)) = \psi(d(W(Tx_n, fx_n, \alpha_n), fv)) \\
&\leq \psi[(1-\alpha_n)d(fx_n, fv) + \alpha_n d(Tx_n, fv)] \\
&= (1-\alpha_n)\psi(d(fx_n, fv)) + \alpha_n \psi(d(Tx_n, Tv)) \\
&\leq (1-\alpha_n)\psi(d(fx_n, fv)) + \alpha_n[\psi(\max\{d(fx_n, fv), d(fx_n, Tx_n), d(fv, Tv)\}) \\
&\quad -\phi(\max\{d(fx_n, fv), d(fx_n, Tx_n), d(fv, Tv)\})] \\
&= (1-\alpha_n)\psi(d(fx_n, fv)) + \alpha_n[\psi(m(f,T)) - \phi(m(f,T))]. \qquad (3.1)
\end{aligned}$$

We have to consider the following two cases.

Case 1: $m(f,T) = d(fx_n, fv)$. Then from (3.1),
$$\begin{aligned}
\psi(d(y_n, u)) &\leq (1-\alpha_n)\psi(d(fx_n, fv)) + \alpha_n[\psi(d(fx_n, fv)) - \phi(d(fx_n, fv))] \\
&= \psi(d(fx_n, fv)) - \alpha_n\psi(d(fx_n, fv)) + \alpha_n\psi(d(fx_n, fv)) - \alpha_n(\phi(d(fx_n, fv))) \\
&= \psi(d(fx_n, fv)) - \alpha_n(\phi(d(fx_n, fv))) \qquad (3.2) \\
&\leq \psi(d(y_{n-1}, u)).
\end{aligned}$$

Case 2: $m(f,T) = d(fx_n, Tx_n)$. Then from (3.1),
$$\begin{aligned}
\psi(d(y_n, u)) &\leq (1-\alpha_n)\psi(d(fx_n, fv)) + \alpha_n[\psi(d(fx_n, Tx_n)) - \phi(d(fx_n, Tx_n))] \\
&= \psi(d(fx_n, fv)) - \alpha_n\psi(d(fx_n, fv)) + \alpha_n(\psi(d(fx_n, Tx_n))) \\
&\quad -\alpha_n(\phi(d(fx_n, Tx_n))) \\
&\leq \psi(d(fx_n, fv)) - \alpha_n\psi[d(fx_n, Tx_n) + d(Tx_n, fv)] + \alpha_n\psi(d(fx_n, Tx_n)) \\
&\quad -\alpha_n(\phi(d(fx_n, Tx_n)))
\end{aligned}$$

$$\begin{aligned}
&= \psi(d(fx_n, fv)) - \alpha_n(\psi(d(Tx_n, fv))) - \alpha_n(\phi(d(fx_n, Tx_n))) \\
&\leq \psi(d(fx_n, fv)) - \alpha_n(\phi(d(fx_n, Tx_n))) \\
&\leq \psi(d(fx_n, fv)) - \alpha_n(\phi[d(fx_n, fv) + d(fv, Tx_n)]) \\
&= \psi(d(fx_n, fv)) - \alpha_n(\phi(d(fx_n, fv))) \qquad (3.3) \\
&\leq \psi(d(y_{n-1}, u)).
\end{aligned}$$

Therefore, in both cases, by monotonic nature of ψ, $d(y_n, u) \leq d(y_{n-1}, u)$. This shows that $\{d(y_n, u)\}$ is a monotone non-increasing sequence of real numbers and tends to a limit $r \geq 0$. So, $\lim_{n\to\infty} d(y_n, u) = r \geq 0$. Now if $r > 0$, then from (3.2) and (3.3), using the lower semi-continuity of ϕ, we obtain

$$\psi(r) \leq \psi(r) - \lim_{n\to\infty} \inf \phi(d(fx_{2n}, fv)) \leq \psi(r) - \phi(r),$$

i.e., $\phi(r) \leq 0$, a contradiction. Therefore, the modified Mann iterative scheme converges to a common fixed point of the maps T and f. □

We remark that Theorem 3.1 (under the weak contractive condition (2.6)) substantially improves certain results of Azam and Shakeel [5], Beg and Abbas [8], Bose and Roychowdhury [11] and Rhoades [34]. Notice that the requirement $\sum \alpha_n = \infty$ on the sequence $\{\alpha_n\}$ in Theorem 3.1 is not needed.

The following is the convergence result for modified Ishikawa iterations in a convex metric space.

Theorem 3.2. *Let T and f be self-maps of a convex metric space such that condition* (2.4) *of Corollary* 2.2 *and the following condition is satisfied:*

$$\psi(d(Tx, Ty)) \leq \psi(d(fx, fy)) - \phi(d(fx, fy)) \qquad (3.4)$$

for all $x, y \in X$, where $\phi \in \Phi$ and $\psi \in \Psi$. If TX or fX is a convex and complete subspace of X, then the modified Ishikawa iterative scheme with $\sum \alpha_n\beta_n = \infty$ converges to a common fixed point of T and f provided that T and f commute at their coincidences.

Proof. From Corollary 2.2, as in the proof of Theorem 3.1, T and f have a coincidence point v and a unique common fixed point u in X such that $Tv = fv = u$. We consider:

$$\begin{aligned}
\psi(d(z_n, u)) &= \psi(d(W(Tp_n, fx_n, \alpha_n), u)) \leq \alpha_n(\psi[d(Tp_n, u) + (1-\alpha_n)d(fx_n, u)]) \\
&\leq \alpha_n\psi(d(Tp_n, Tv)) + (1-\alpha_n)\psi(d(fx_n, u)) \\
&\leq \alpha_n[\psi(d(fp_n, fv)) - \phi(d(fp_n, fv))] + (1-\alpha_n)\psi(d(fx_n, u)) \\
&= \alpha_n\psi(d(fp_n, u)) - \alpha_n\phi(d(fp_n, u)) + (1-\alpha_n)\psi(d(fx_n, u)) \\
&= \alpha_n\psi(d(W(Tx_n, fx_n, \beta_n), u)) - \alpha_n\phi(d(fp_n, u)) + (1-\alpha_n)\psi(d(fx_n, u)) \\
&\leq \alpha_n(\psi[\beta_n d(Tx_n, u) + (1-\beta_n)d(fx_n, u)]) - \alpha_n\phi(d(fp_n, u)) + \\
&\quad (1-\alpha_n)\psi(d(fx_n, u)) \\
&= \alpha_n\beta_n(\psi(d(Tx_n, Tu))) + \alpha_n(1-\beta_n)\psi(d(fx_n, u)) - \alpha_n\phi(d(fp_n, u)) \\
&\quad +(1-\alpha_n)\psi(d(fx_n, u))
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n\beta_n[\psi(d(fx_n,u)) - \phi(d(fx_n,u))] + \alpha_n(1-\beta_n)\psi(d(fx_n,u)) \\
&\quad -\alpha_n\phi(d(fp_n,u)) + (1-\alpha_n)\psi(d(fx_n,u)) \\
&= \alpha_n\beta_n\psi(d(fx_n,u)) - \alpha_n\beta_n\phi(d(fx_n,u)) + \alpha_n\psi(d(fx_n,u)) \\
&\quad -\alpha_n\beta_n\psi(d(fx_n,u)) - \alpha_n\phi(d(fp_n,u)) + \psi(d(fx_n,u)) - \alpha_n\psi(d(fx_n,u)) \\
&\leq \psi(d(fx_n,u)) - \alpha_n\beta_n\phi(d(fx_n,u)) - \alpha_n\phi(d(fp_n,u)) \leq \psi(d(fx_n,u)).
\end{aligned}$$

Therefore, the sequence $\{d(z_n,u)\}$ is nonincreasing nonnegative sequence which converges to the limit $r \geq 0$. Suppose that $r > 0$. Then for a fixed integer N, we have

$$\sum_{n=N}^{\infty} \alpha_n\beta_n\phi(r) \leq \sum_{n=N}^{\infty} \alpha_n\beta_n\phi(d(z_n,u)) \leq \sum_{n=N}^{\infty} (d(z_n,u) - d(z_{n+1},u)) \leq d(z_N,u).$$

This is a contradiction to $\sum \alpha_n\beta_n = \infty$. Hence, the iterative scheme $\{z_n\}$ converges to the common fixed point u. This completes the proof. □

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