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INTRODUCING A NONLINEAR CONNECTION IN A LAGRANGIAN SUPERMECHANICAL SYSTEM

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Abstract. In this paper, we study the concept of a Lagrangian supermechanical system. The dynamical system of a Lagrangian supermechanical system defined by a second order ordinary differential equation which introduce a superspray S as a solution of the Euler-Lagrange equation. There is a nonlinear connection N, induced by this superspray. Since every superspray on a supermanifold determines a nonlinear connections and vice versa, so we have a sequence of supersprays via nonlinear connections associated with them. Using this idea and under the conditions on Lagrangian superfunctions, we will find a nonlinear connection $\overset{1}{N}$ such that its superspray determines the above nonlinear connection N.

1. INTRODUCTION

The concept of a nonlinear connection (N-connection) was introduced in component form in a number of works by Cartan [4], Kawaguchi [8, 9] and Ehresmann [6]. But the first global definition is due to Barthel [3]. The geometry of N-connection in superspaces and applications are considered in detail in [1, 2] and [11]. In [1, 2] A. Bejancu introduced a nonlinear connection (H-connection) structure in the frame of DeWitt approach to supermanifolds, by considering the particular case of superbundles with typical fibers parameterized by noncommutative coordinates. In [11], Vacaru gave a general definition of locally anistropic superspaces. He formulated the theory of vector and tangent superbundles provided with nonlinear and distinguished connections and metric structures. He also defined the concepts of Generalized Lagrange superspaces.

In this paper we pay attention to the method of constructing a nonlinear connection in a Lagrangian supermechanical system. A Lagrangian supermechanical system is a triple $(\mathcal{M}, L, V)$ where $(\mathcal{M}, L)$ is a Lagrange superspace and V is a vertical supervector field, which is called the external force field of the system. The dynamical system of a Lagrangian supermechanical system defined by a second order ordinary differential equation. We show that there is a superspray on $T\mathcal{M}$ such that its coefficients obtained by this equation.

As we know, there is a one-to-one correspondence between supersprays and nonlinear connections on a supermanifold $\mathcal{M}$, so there is a nonlinear connection N on $\mathcal{M}$ (so called the canonical nonlinear connection) such that its coefficients obtained by differentiating of the coefficients of the superspray with respect to y^i and θ^α.

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In [7], it was shown, every superspray on a supermanifold determines a nonlinear connections and vice versa. By using this method we will have a sequence of supersprays via nonlinear connections associated with them. Using this idea and under the conditions on Lagrangian superfunctions, we will find a nonlinear connection $\overset{1}{N}$ in a Lagrangian supermechanical system such that its superspray determines the above nonlinear connection N.

2. PRELIMINARY

Let $\mathcal{M}$ be a supermanifold of dimensional (m,n). A vector superbundle $\mathcal{E}$ over a supermanifold $\mathcal{M}$ ([12], [13]) with total superspace E, standard fibre F and surjective projection $\pi_E\colon E \mapsto M$ is defined as in the case of ordinary manifolds. The idea of construction is by using the change of coordinate functions of supermanifold and transition functions of the superbundle. Particular example of this construction is the supermanifold $T\mathcal{M}$. The tangent superbundle $T\mathcal{M}$ over a supermanifold $\mathcal{M}$ is constructed in a usual manner. If $\mathcal{M}$ has dimension (m,n) and coordinates $(x;\eta)$, then $T\mathcal{M}$ has dimension $(2m,2n)$ and local coordinates $(x,y;\eta,\theta)$ which change, [5], according to the rule

$$\begin{aligned} &x'^i = x'^i(x^1,\dots,x^m;\eta^1,\dots,\eta^n), \quad \eta'^\alpha = \eta'^\alpha(x^1,\dots,x^m;\eta^1,\dots,\eta^n),\\ &y'^j = \sum_{i=1}^{m}\frac{\partial x'^j}{\partial x^i}y^i + \sum_{\alpha=1}^{n}\frac{\partial x'^j}{\partial \eta^\alpha}\theta^\alpha, \qquad \theta'^\beta = \sum_{i=1}^{m}\frac{\partial \eta'^\beta}{\partial x^i}y^i + \sum_{\alpha=1}^{n}\frac{\partial \eta'^\beta}{\partial \eta^\alpha}\theta^\alpha. \end{aligned} \tag{2.1}$$

The transition functions (2.1) give rise to a supermanifold that will be denoted by $T\mathcal{M}$ and is called the tangent supermanifold of $\mathcal{M}$. For the tangent superbundle the number of x's and y's are the same (y plays the role of $\dot{x}$) and the number of η's and θ's is also the same.

Locally a nonlinear connection in $\mathcal{E}$ is given (see [10], [13]) by its coefficients

$$N_i^j(x,y,\eta,\quad\theta),\quad N_i^\beta(x,y,\eta,\theta),\quad N_\alpha^j(x,y,\eta,\theta),\quad N_\alpha^\beta(x,y,\eta,\theta).$$

Under (2.1) we have the following transformation laws of the local coefficients of N:

$$\frac{\partial x'^k}{\partial x^i}N_k'^j = N_i^k\frac{\partial x'^j}{\partial x^k} - N_i^\alpha\frac{\partial x'^j}{\partial \eta^\alpha} - \frac{\partial^2 x'^j}{\partial x^i\partial x^k}y^k - \frac{\partial^2 x'^j}{\partial x^i\partial\eta^\alpha}\theta^\alpha, \tag{2.2}$$

$$\frac{\partial x'^k}{\partial x^i}N_k'^\beta = N_i^j\frac{\partial \eta'^\beta}{\partial x^j} + N_i^\alpha\frac{\partial \eta'^\beta}{\partial \eta^\alpha} - \frac{\partial^2 \eta'^\beta}{\partial x^i\partial x^k}y^k - \frac{\partial^2 \eta'^\beta}{\partial x^i\partial\eta^\alpha}\theta^\alpha, \tag{2.3}$$

$$\frac{\partial \eta'^\gamma}{\partial \eta^\alpha}N_\gamma'^i = N_\alpha^j\frac{\partial x'^i}{\partial x^j} - N_\alpha^\beta\frac{\partial x'^i}{\partial \eta^\beta} - \frac{\partial^2 x'^i}{\partial\eta^\alpha\partial x^j}y^j - \frac{\partial^2 x'^i}{\partial\eta^\alpha\partial\eta^\beta}\theta^\beta \tag{2.4}$$

and

$$\frac{\partial \eta'^\gamma}{\partial \eta^\alpha}N_\gamma'^\beta = N_\alpha^i\frac{\partial \eta'^\beta}{\partial x^j} + N_\alpha^\gamma\frac{\partial \eta'^\beta}{\partial \eta^\gamma} - \frac{\partial^2 \eta'^\beta}{\partial\eta^\alpha\partial x^i}y^i - \frac{\partial^2 \eta'^\beta}{\partial\eta^\alpha\partial\eta^\gamma}\theta^\gamma. \tag{2.5}$$

Let L be a Lagrangian defined on $T\mathcal{M}$. A Lagrange superspace is a pair $(\mathcal{M},L)$ formed by a supermanifold $\mathcal{M}$ and a regular Lagrangian $L(x,y,\eta,\theta)$, (see [11]).

The dynamics of a system $(T\mathcal{M},\omega,L)$ associated to a regular Lagrangian L is given by a supervector field $X\in\chi(T\mathcal{M})$ satisfying the equation

$$i_X\omega = -dL, \tag{2.6}$$

where $\omega = dd_JL$ and the distinguished tensor field $g(x,y;\eta,\theta)$ on $\mathcal{M}$ is defined as

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i\partial y^j},\quad g_{i\beta} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i\partial\theta^\beta},\quad g_{\alpha j} = \frac{1}{2}\frac{\partial^2 L}{\partial\theta^\alpha\partial y^j},\quad g_{\alpha\beta} = \frac{1}{2}\frac{\partial^2 L}{\partial\theta^\alpha\partial\theta^\beta}. \tag{2.7}$$

In such case, there exists an even supervector field, the Euler- Lagrange supervector field on $T\mathcal{M}$ such that it is the solution of the dynamical equation (2.6) (see [10]). In terms of local coordinates $(x^i, y^i; \eta^\alpha, \theta^\alpha)$, the Euler-Lagrange equations of L are

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial L}{\partial y^i} - \dfrac{\partial L}{\partial x^i} = 0, & y^i = \dfrac{dx^i}{dt} \\ \dfrac{d}{dt}\dfrac{\partial L}{\partial \theta^\alpha} - \dfrac{\partial L}{\partial \eta^\alpha} = 0, & \theta^\alpha = \dfrac{d\eta^\alpha}{dt}. \end{cases} \tag{2.8}$$

From these equations we conclude that

$$g_{ki}\frac{d^2x^k}{dt^2} - (-1)^{|L|}g_{\alpha i}\frac{d^2\eta^\alpha}{dt^2} = \frac{1}{2}\left(\frac{\partial L}{\partial x^i} - \frac{\partial^2 L}{\partial x^k \partial y^i}y^k + (-1)^{|L|}\frac{\partial^2 L}{\partial \eta^\alpha \partial y^i}\theta^\alpha\right) \tag{2.9}$$

and

$$g_{i\alpha}\frac{d^2x^i}{dt^2} + (-1)^{|L|}g_{\beta\alpha}\frac{d^2\eta^\beta}{dt^2} = \frac{1}{2}\left(\frac{\partial L}{\partial \eta^\alpha} - \frac{\partial^2 L}{\partial x^i \partial \theta^\alpha}y^i - (-1)^{|L|}\frac{\partial^2 L}{\partial \eta^\beta \partial \theta^\alpha}\theta^\beta\right). \tag{2.10}$$

We assumed that L is a regular Lagrangian, this means that $\begin{pmatrix} g_{ki} & g_{\beta i} \\ g_{k\alpha} & g_{\alpha\beta} \end{pmatrix}$ has an inverse and we denote it by $\begin{pmatrix} g^{ik} & g^{k\beta} \\ g^{\beta k} & g^{\alpha\beta} \end{pmatrix}$, so

$$\begin{aligned} \frac{d^2x^k}{dt^2} &= \frac{1}{2}\left(g^{ik}A_{ik} + g^{k\beta}B_{\alpha\beta}\right), \\ \frac{d^2\eta^\beta}{dt^2} &= -\frac{(-1)^{|L|}}{2}\left(g^{\beta k}A_{ik} + g^{\alpha\beta}B_{\alpha\beta}\right), \end{aligned}$$

where A_{ik} and $B_{\alpha\beta}$ are the right-hand side of the equations (2.9) and (2.10), respectively.

These equations are equivalent with the system of differential equations of second order

$$\frac{d^2x^k}{dt^2} + 2\bar{G}^k = 0, \quad \frac{d^2\eta^\beta}{dt^2} + 2\bar{G}^\beta = 0, \quad y^i = \frac{dx^i}{dt}, \quad \theta^\alpha = \frac{d\eta^\alpha}{dt}$$

where

$$\bar{G}^k = -\frac{1}{2}\left(g^{ik}A_{ik} + g^{k\beta}B_{\alpha\beta}\right), \quad \bar{G}^\beta = \frac{(-1)^{|L|}}{2}\left(g^{\beta k}A_{ik} + g^{\alpha\beta}B_{\alpha\beta}\right).$$

With respect to these coefficients, the superspray associated to Lagrangian L is

$$\bar{S} = y^i\frac{\partial}{\partial x^i} + \theta^\alpha\frac{\partial}{\partial \eta^\alpha} - 2\bar{G}^i\frac{\partial}{\partial y^i} - 2\bar{G}^\alpha\frac{\partial}{\partial \theta^\alpha}. \tag{2.11}$$

In [7], it was shown, there is a one-to-one correspondence between the susprays and nonlinear connections on a supermanifold $\mathcal{M}$, i.e. if we have a superspray then we have a nonlinear connection and for each nonlinear connection we have a superspray. The coefficients of the nonlinear connection associated to a superspray S obtained by differentiating of the coefficients of the superspray with respect to y^i and θ^α. So the nonlinear connection $\bar{N}$, associated to the superspray $\bar{S}$, has the coefficients

$$\bar{N}^i_j := \frac{\partial \bar{G}^i}{\partial y^j}, \quad \bar{N}^i_\alpha := \frac{\partial \bar{G}^i}{\partial \theta^\alpha}, \quad \bar{N}^\alpha_i := \frac{\partial \bar{G}^\alpha}{\partial y^i}, \quad \bar{N}^\alpha_\beta := \frac{\partial \bar{G}^\alpha}{\partial \theta^\beta}.$$

Example 2.1. Let $g = g(x, y, \eta, \theta)$ be a Riemann metric on $T\mathcal{M}$ with coefficients $g_{ij}, g_{i\alpha}$ and $g_{\alpha\beta}$ as superfunctions on $T\mathcal{M}$, then

$$L = g_{ij}y^i y^j + 2g_{i\alpha}y^i\theta^\alpha + g_{\alpha\beta}\theta^\alpha\theta^\beta$$

is a Lagrangian superfunction on $T\mathcal{M}$. By differentiating of L with respect to y^k and θ^γ, we have

$$\frac{\partial L}{\partial y^k} = 2g_{ik}y^i + \frac{\partial g_{ij}}{\partial y^k}y^i y^j + 2g_{i\alpha}\theta^\alpha + 2\frac{\partial g_{i\alpha}}{\partial y^k}y^i\theta^\alpha + \frac{\partial g_{\alpha\beta}}{\partial y^i}\theta^\alpha\theta^\beta$$

and

$$\frac{\partial L}{\partial \theta^\gamma} = \frac{\partial g_{ij}}{\partial \theta^\gamma}y^i y^j + 2\frac{\partial g_{i\alpha}}{\partial \theta^\gamma}y^i\theta^\alpha + 2(-1)^{g_{i\alpha}}g_{i\gamma}y^i + \left\{\frac{\partial g_{\alpha\beta}}{\partial \theta^\gamma}\theta^\alpha\theta^\beta + 2(-1)^{g_{\gamma\beta}}g_{\gamma\beta}\theta^\beta\right\}.$$

3. LAGRANGIAN SUPERMECHANICAL SYSTEMS

Definition 3.1. A Lagrangian supermechanical system is a triple $\sum = (\mathcal{M}, L, V)$ where $(\mathcal{M}, L)$ is a Lagrange superspace and $V = V^i\dfrac{\partial}{\partial y^i} + V^\alpha\dfrac{\partial}{\partial \theta^\alpha}$ is a vertical supervector field, which is called the *external force field of the system.*

The evolution equations of the Lagrangian supermechanical system $\sum$ are the following Lagrange equations:

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial L}{\partial y^i} - \dfrac{\partial L}{\partial x^i} = g_{ij}V^j + g_{i\alpha}V^\alpha, & y^i = \dfrac{dx^i}{dt} \\ \dfrac{d}{dt}\dfrac{\partial L}{\partial \theta^\alpha} - \dfrac{\partial L}{\partial \eta^\alpha} = g_{\alpha i}V^i + g_{\alpha\beta}V^\beta, & \theta^\alpha = \dfrac{d\eta^\alpha}{dt}. \end{cases} \tag{3.1}$$

These equations are equivalent with the system of differential equations of second order

$$\frac{d^2x^k}{dt^2} + 2\bar{G}^k = \frac{1}{2}V^k, \quad \frac{d^2\eta^\beta}{dt^2} + 2\bar{G}^\beta = \frac{1}{2}V^\beta$$

where $\bar{G}^i$ and $\bar{G}^\alpha$ are local coefficient of the superspray (2.11) of the Lagrange superspace (M, L). When we change the local coordinate system $(x; \eta)$ in the base supermanifold $\mathcal{M}$, the corresponding standard local coordinate system $(x, y; \eta, \theta)$ in $T\mathcal{M}$ changes and the superfunctions

$$2G^k = 2\bar{G}^k - \frac{1}{2}V^k, \quad 2G^\beta = 2\bar{G}^\beta - \frac{1}{2}V^\beta \tag{3.2}$$

change subject to the transformation law. Because V^i and V^α are components of a supervector field, so these are local coefficients of the superspray

$$S = y^i\frac{\partial}{\partial x^i} + \theta^\alpha\frac{\partial}{\partial \eta^\alpha} - 2G^i\frac{\partial}{\partial y^i} - 2G^\alpha\frac{\partial}{\partial \theta^\alpha}.$$

An easy computation shows that

$$i_V\omega = \left(2g_{ij}V^j - (-1)^{|L|}2g_{i\alpha}V^\alpha\right)dx^i + \left(-(-1)^{|L|}2g_{i\alpha}V^i - 2g_{\beta\alpha}V^\beta\right)d\eta^\alpha.$$

Since $i_{\bar{S}}\omega = -dL$, so we can conclude that

Theorem 3.1. *The superspray*

$$S = \bar{S} + \frac{1}{2}\left(V^i \frac{\partial}{\partial y^i} + V^\alpha \frac{\partial}{\partial \theta^\alpha}\right) \tag{3.3}$$

is the unique supervector field on $T\mathcal{M}$ satisfying the equation

$$i_S\omega = -dL + \left(2g_{ij}V^j - (-1)^{|L|}2g_{i\alpha}V^\alpha\right)dx^i + \left(-(-1)^{|L|}2g_{i\alpha}V^i - 2g_{\beta\alpha}V^\beta\right)d\eta^\alpha.$$

Now, as explained in the previous section, we have a nonlinear connection N on $\mathcal{M}$, associated to the superspray S such that its coefficients obtained by differentiating of the coefficients of the superspray with respect to y^i and θ^α. A straightforward computation shows that N has the local coefficients given by

$$\begin{aligned} N^i_j &= \bar{N}^i_j - \frac{1}{4}\frac{\partial V^i}{\partial y^j}, \qquad N^i_\alpha = \bar{N}^i_\alpha - \frac{1}{4}\frac{\partial V^i}{\partial \theta^\alpha} \\ N^\alpha_i &= \bar{N}^\alpha_i - \frac{1}{4}\frac{\partial V^\alpha}{\partial y^j}, \quad N^\alpha_\beta = \bar{N}^\alpha_\beta - \frac{1}{4}\frac{\partial V^\alpha}{\partial \theta^\beta}. \end{aligned} \tag{3.4}$$

It is not difficult to show that these superfunctions satisfy the transformation rules (2.2) - (2.5) hence define a nonlinear connection (so called the canonical nonlinear connection).

4. GENERATING THE NONLINEAR CONNECTIONS

As we mentioned earlier, there is a correspondence between a superspray and the associated nonlinear connection. Thus it seems that for each superspray S we can give a sequence of nonlinear connections [10]. To see this, start with a superspray $\overset{1}{S}$ on $\mathcal{M}$ with coefficients $\overset{1}{G}{}^i, \overset{1}{G}{}^\alpha$ and let $\overset{1}{N}$ be the nonlinear connection of the superspray $\overset{1}{S}$. This nonlinear connection $\overset{1}{N}$, determines a superspray $\overset{2}{S}$ with local coefficients

$$\overset{2}{G}{}^i(x,y;\eta,\theta) = \frac{1}{2}y^j\overset{1}{N}{}^i_j + \frac{1}{2}\theta^\alpha\overset{1}{N}{}^i_\alpha, \quad \overset{2}{G}{}^\alpha(x,y;\eta,\theta) = \frac{1}{2}y^j\overset{1}{N}{}^\alpha_j + \frac{1}{2}\theta^\beta\overset{1}{N}{}^\alpha_\beta,$$

where $\overset{1}{N}{}^i_j, \overset{1}{N}{}^i_\alpha, \overset{1}{N}{}^\alpha_j$ and $\overset{1}{N}{}^\alpha_\beta$ are the local coefficients of $\overset{1}{N}$. By differentiating of the coefficients of the superspray $\overset{2}{S}$ with respect to y^i and θ^α we have the following superfunctions

$$\overset{2}{N}{}^i_j := \frac{\partial \overset{2}{G}{}^i}{\partial y^j}, \quad \overset{2}{N}{}^i_\alpha := \frac{\partial \overset{2}{G}{}^i}{\partial \theta^\alpha}, \quad \overset{2}{N}{}^\alpha_i := \frac{\partial \overset{2}{G}{}^\alpha}{\partial y^i}, \quad \overset{2}{N}{}^\alpha_\beta := \frac{\partial \overset{2}{G}{}^\alpha}{\partial \theta^\beta} \tag{4.1}$$

which determine a nonlinear connection $\overset{2}{N}$. By this method we can obtain a sequence of nonlinear connections $(\overset{k}{N})_{k\in N}$ on $T\mathcal{M}$.

Now consider the nonlinear connection N with coefficients (3.4) constructed by the superspray (3.3). We want to find a nonlinear connection $\overset{1}{N}$ such that its superspray determines the nonlinear connection $\overset{2}{N} = N$ with local coefficients (3.4). To do so, we need to reformulate the coefficients of $\overset{2}{N}$. Since N constructed by S, so $N^i_j = \dfrac{\partial G^i}{\partial y^j}$, on the other

hand $N = \overset{2}{N}$, i.e. $\overset{2}{N}{}^i_j = \frac{1}{2}\frac{\partial}{\partial y^j}\Big\{y^k \overset{1}{N}{}^i_k + \theta^\alpha \overset{1}{N}{}^i_\alpha\Big\}$. Thus there is a superfunction $\mu^i(x,\eta)$ on $\mathcal{M}$ such that we have

$$y^j \overset{1}{N}{}^i_j + \theta^\alpha \overset{1}{N}{}^i_\alpha = 2G^i + \mu^i(x,\eta). \tag{4.2}$$

Similarly there is a superfunction $\mu^\alpha(x,\eta)$ on $\mathcal{M}$ such that

$$y^j \overset{1}{N}{}^\alpha_j + \theta^\beta \overset{1}{N}{}^\alpha_\beta = 2G^\alpha + \mu^\alpha(x,\eta). \tag{4.3}$$

The relation between a superspray (S,G) in a Lagrangian supermechanical system and $(\bar{S},\bar{G})$, a superspray in a Lagrange superspace can help us to simplify the above equations. The reason for this is that we have more information about the Lagrangian superspaces and the canonical superspray associated to it with respect to the canonical superspray associated to a Lagrangian supermechanical system. So with using this relation which asserted in (3.2), we can rewrite (4.2) and (4.3) as

$$\begin{cases} y^j \overset{1}{N}{}^i_j + \theta^\alpha \overset{1}{N}{}^i_\alpha = 2\bar{G}^i - \frac{1}{2}V^i + \mu^i(x,\eta), \\ y^j \overset{1}{N}{}^\alpha_j + \theta^\beta \overset{1}{N}{}^\alpha_\beta = 2\bar{G}^\alpha - \frac{1}{2}V^\alpha + \mu^\alpha(x,\eta). \end{cases} \tag{4.4}$$

The superfunctions $\overset{1}{N}{}^i_j, \overset{1}{N}{}^i_\alpha, \overset{1}{N}{}^\alpha_j$ and $\overset{1}{N}{}^\alpha_\beta$, solutions of the above equations, determine a nonlinear connection on $\mathcal{M}$. Although we can not find the complete solutions of the equations, but we give some partial solutions.

It should be noted that according to the above relationship, (4.4), some sentences of the coefficients of $\overset{1}{N}$ have been introduced in the coefficients of G. In other words, we can use it to introduce the coefficients of the nonlinear connection. This can be done using proper Lagrangians. So we try to introduce an appropriate Lagrangian to achieve good results.

Theorem 4.1. *Let us consider a Lagrangian supermechanical system $\sum = (\mathcal{M}, L, V)$ where $(\mathcal{M}, L)$ is a Lagrange superspace, L is a 2-homogeneous Lagrangian superfunction of variables y^i and θ^α and the coefficients of the external force, V^i and V^α, are homogeneous superfunctions of degree 2. There is a nonlinear connection that will produce the canonical nonlinear connection.*

Proof. The coefficients of the canonical superspray associated to this superspace are homogeneous superfunctions of degree two, see (3.3) of the Theorem 3.1, so the nonlinear connection associated to this superspray has 1-homogeneous superfunction, see (3.4). From the homogeneity property of the coefficients of the superspray and the nonlinear connection, we can conclude that all superfunctions in the first line of the above equation, (4.4), are homogeneous except $\mu^i(x,\eta)$, thus it should be zero. By using the same reasoning, we have $\mu^\alpha(x,\eta) = 0$.

Now, consider the nonlinear connection $\overset{1}{N} = N$ defined in (3.4). Since the coefficients of N are homogeneous functions of degree 1, thus

$$\begin{aligned} \overset{2}{N}{}^i_\alpha &= \frac{1}{2}\frac{\partial}{\partial\theta^\alpha}\Big\{y^k \overset{1}{N}{}^i_k + \theta^\beta \overset{1}{N}{}^i_\beta\Big\} = \frac{1}{2}\frac{\partial}{\partial\theta^\alpha}\{y^k N^i_k + \theta^\beta N^i_\beta\} \\ &= \frac{1}{2}\Big\{y^k \frac{\partial N^i_k}{\partial\theta^\alpha} + N^i_\beta - \theta^\beta \frac{\partial N^i_\beta}{\partial\theta^\alpha}\Big\}. \end{aligned}$$

Using the property $\frac{\partial N^i_\beta}{\partial\theta^\alpha}=\frac{\partial^2 G^i}{\partial\theta^\alpha\partial\theta^\beta}=-\frac{\partial^2 G^i}{\partial\theta^\beta\partial\theta^\alpha}$ then we have $\overset{2}{N}{}^i_\alpha=N^i_\alpha$. We can easily show that the same result is established for the remaining coefficients of $\overset{2}{N}$. So $\overset{2}{N}=N$. In other words, the sequence of nonlinear connections associated to $\overset{1}{N}$ is constant. □

Theorem 4.2. *Let $L=L_i(x,y,\eta,\theta)y^i+L_\alpha(x,y,\eta,\theta)\theta^\alpha$ be a regular 2-homogeneous Lagrange superfunction on $T\mathcal{M}$ and the coefficients of the external force, V^i and V^α, are homogeneous superfunctions of degree 2. Then there is a nonlinear connection that will produce the canonical nonlinear connection.*

Proof. Our goal is to introduce a nonlinear connection of degree 1, i.e. a nonlinear connection which has coefficients as 1-homogeneous superfunctions. A similar way applied in Theorem 4.1 shows that $\mu^i=0$ and $\mu^\alpha=0$. The metric tensor associated to this Lagrange superfunction has the following coefficients

$$\begin{aligned}
g_{ij}&=\frac{1}{2}\left(\frac{\partial^2 L_k}{\partial y^i\partial y^j}y^k+\frac{\partial L_i}{\partial y^j}+\frac{\partial^2 L_\alpha}{\partial y^i\partial y^j}\theta^\alpha\right),\\
g_{i\alpha}&=\frac{1}{2}\left(\frac{\partial^2 L_\beta}{\partial y^i\partial\theta^\alpha}\theta^\beta+\frac{\partial L_i}{\partial\theta^\alpha}-(-1)^{|L|}\frac{\partial L_\alpha}{\partial y^i}+\frac{\partial^2 L_k}{\partial\theta^\alpha\partial y^i}y^k\right),\\
g_{\alpha\beta}&=\frac{1}{2}\left(\frac{\partial^2 L_i}{\partial\theta^\alpha\partial\theta^\beta}y^i+\frac{\partial^2 L_\mu}{\partial\theta^\alpha\partial\theta^\beta}\theta^\mu+(-1)^{|L|}\frac{\partial L_\alpha}{\partial\theta^\beta}-(-1)^{|L|}\frac{\partial L_\beta}{\partial\theta^\alpha}\right).
\end{aligned}$$

Let $\begin{pmatrix} g^{ik} & g^{k\beta}\\ g^{\beta k} & g^{\alpha\beta}\end{pmatrix}$ be the inverse of $\begin{pmatrix} g_{ki} & g_{\beta i}\\ g_{k\alpha} & g_{\alpha\beta}\end{pmatrix}$. If we choose the nonlinear connection N with the coefficients

$$\begin{aligned}
\overset{1}{N}{}^j_m&=\frac{1}{2}g^{jm}\left(\frac{\partial^2 L_k}{\partial x^m\partial y^j}y^k+\frac{\partial L_j}{\partial x^m}+\frac{\partial^2 L_\alpha}{\partial x^m\partial y^j}\theta^\alpha-\frac{\partial L_m}{\partial x^j}\right)\\
&\quad+\frac{1}{2}g^{m\gamma}\left(\frac{\partial^2 L_k}{\partial x^m\partial\theta^\gamma}y^k+\frac{\partial^2 L_\mu}{\partial x^m\theta^\gamma}\theta^\mu-(-1)^{|L|}\frac{\partial L_\gamma}{\partial x^m}-\frac{\partial L_k}{\partial\eta^\gamma}y^k\right)-\frac{1}{4}\frac{\partial V^j}{\partial y^m},\\
\overset{1}{N}{}^j_\gamma&=\frac{1}{2}g^{jm}\left(\frac{\partial^2 L_k}{\partial\eta^\alpha\partial y^j}y^k+\frac{\partial L_j}{\partial\eta^\alpha}+\frac{\partial^2 L_\beta}{\partial\eta^\alpha\partial y^j}\theta^\beta+(-1)^{|L|}\frac{\partial L_\alpha}{\partial x^j}\right)\\
&\quad+\frac{1}{2}g^{j\gamma}\left(-(-1)^{|L|}\left\{\frac{\partial^2 L_k}{\partial\eta^\alpha\partial\theta^\gamma}y^k+\frac{\partial^2 L_\mu}{\partial\eta^\alpha\theta^\gamma}\theta^\mu\right\}+\frac{\partial L_\gamma}{\partial\eta^\alpha}+\frac{\partial L_\alpha}{\partial\eta^\gamma}\right)-\frac{1}{4}\frac{\partial V^j}{\partial\theta^\gamma},
\end{aligned}$$

then we have $y^m\overset{1}{N}{}^j_m+\theta^\alpha\overset{1}{N}{}^j_\gamma=2G^j$. Also, if

$$\begin{aligned}
\overset{1}{N}{}^\alpha_k&=\frac{(-1)^{|L|}}{2}g^{\beta k}\left(-\left\{\frac{\partial^2 L_j}{\partial x^k\partial y^i}y^j+\frac{\partial L_i}{\partial x^k}+\frac{\partial^2 L_\mu}{\partial x^k\partial y^i}\theta^\mu\right\}+\frac{\partial L_k}{\partial x^i}\right)\\
&\quad+\frac{(-1)^{|L|}}{2}g^{\alpha\beta}\left(-\left\{-\frac{\partial^2 L_j}{\partial x^k\partial\theta^\alpha}y^j+\frac{\partial^2 L_\mu}{\partial x^k\theta^\alpha}\theta^\mu-(-1)^{|L|}\frac{\partial L_\alpha}{\partial x^k}\right\}+\frac{\partial L_k}{\partial\eta^\alpha}\right)-\frac{1}{4}\frac{\partial V^\alpha}{\partial y^k},\\
\overset{1}{N}{}^\alpha_\beta&=\frac{1}{2}g^{\beta k}\left(\frac{\partial^2 L_j}{\partial\eta^\beta\partial y^i}y^j+\frac{\partial L_i}{\partial\eta^\beta}+\frac{\partial^2 L_\mu}{\partial\eta^\beta\partial y^i}+(-1)^{|L|}\frac{\partial L_\mu}{\partial x^i}\right)\\
&\quad+\frac{1}{2}g^{\alpha\beta}\left(-\frac{\partial^2 L_j}{\partial\eta^\beta\partial\theta^\alpha}y^j-\frac{\partial^2 L_\mu}{\partial\eta^\beta\theta^\alpha}\theta^\mu+(-1)^{|L|}\frac{\partial L_\alpha}{\partial\eta^\beta}+(-1)^{|L|}\frac{\partial L_\beta}{\partial\eta^\alpha}\right)-\frac{1}{4}\frac{\partial V^\alpha}{\partial\theta^\beta}
\end{aligned}$$

thus we have $y^k \overset{1}{N}{}^{\alpha}_{k} + \theta^{\beta} \overset{1}{N}{}^{\alpha}_{\beta} = 2G^{\alpha}$. So we were able to introduce a nonlinear connection in a Lagrangian supermechanical system to produce the canonical nonlinear connection. □

As an example for this case, suppose that $L_i = \bar{g}_{ij} y^j + \bar{g}_{i\alpha} \theta^{\alpha}$ and $L_{\alpha} = \bar{g}_{\alpha i} y^i + \bar{g}_{\alpha\beta} \theta^{\beta}$ where $\bar{g}_{ij}, \bar{g}_{i\alpha}$ and $\bar{g}_{\alpha\beta}$ are the coefficients of a Riemannian metric on $\mathcal{M}$. Then L is regular and the coefficients of $\overset{1}{N}$ can be introduced clearly.

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